Symmetries shape the current in ratchets induced by a bi-harmonic force. Supplementary Material

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Let us analyze the following evolution equations $E[x(t), f(t)] = 0$ for the variables $x(t)$ (position) and $u(t)$ (velocity) of a relativistic particle of mass $M > 0$

$$\begin{aligned} M\frac{du}{dt} &= -f(t)(1-u^2)^{3/2} - \gamma u(1-u^2), \\ \frac{dx}{dt} &= u(t), \qquad u(0) = u_0, \quad x(0) = x_0, \end{aligned} \tag{1}$$

where x_0 and u_0 are the initial conditions, $\gamma > 0$ represents the damping coefficient and $f(t)$ is a T-periodic driving force [1]. Notice that defining the momentum

$$P(t) = \frac{Mu(t)}{\sqrt{1-u^2(t)}}, \tag{2}$$

we can transform Eq. (1) into the linear equation

$$\frac{dP}{dt} = -\beta P - f(t), \tag{3}$$

where $\beta = \gamma/M$, whose solution is given by

$$P(t) = P(0)e^{-\beta t} - \int_0^t dz f(z) e^{-\beta(t-z)}. \tag{4}$$

Equation (1) is invariant under *time shift* ($\mathcal{S} : t \mapsto t + T/2$) along with the change $x \mapsto -x$, provided $(\mathcal{S}f)(t) = f(t+T/2) = -f(t)$. The bi-harmonic force

$$f(t) = \epsilon_1 \cos(q\omega t + \phi_1) + \epsilon_2 \cos(p\omega t + \phi_2), \tag{5}$$

preserves this symmetry if, both, p and q are odd integer numbers, so in this case the average velocity

$$v = \lim_{t\to+\infty} \frac{1}{t}\int_0^t u(\tau)\, d\tau, \tag{6}$$

is zero. In contrast, if $p+q$ is odd and p and q are coprimes, a nonzero average current can appear. For the sake of simplicity we will take $p = 2$ and $q = 1$ in Eq. (5) [2]. Then the solution to (4) for the chosen force (5) will be

$$\begin{aligned} P(t) &= \tilde{P}_0 \exp(-\beta t) - \frac{\epsilon_1}{\sqrt{\beta^2+\omega^2}} \cos(\omega t + \phi_1 - \chi_1) \\ &- \frac{\epsilon_2}{\sqrt{\beta^2+4\omega^2}} \cos(2\omega t + \phi_2 - \chi_2), \end{aligned} \tag{7}$$

with $\tilde{P}_0 = P(0) + (\epsilon_1/\sqrt{\beta^2+\omega^2})\cos(\phi_1 - \chi_1) + (\epsilon_2/\sqrt{\beta^2+4\omega^2})\cos(\phi_2 - \chi_2)$, $\chi_1 = \arctan(\omega/\beta)$, and $\chi_2 = \arctan(2\omega/\beta)$. From (2), one obtains

$$u(t) = \sum_{k=0}^{\infty} \frac{(-1)^k (1/2)_k}{k! M^{2k+1}} [P(t)]^{2k+1}, \tag{8}$$

where $(1/2)_k \equiv (1/2)(1/2+1)\cdots(1/2+k-1)$. From (6) and (8) it follows that the time-average velocity, v, cannot be expressed as a function of the odd moments of $f(t)$, unless $P(t)$ is proportional to $f(t)$. Indeed, it is only in the overdamped case [in which the inertial term in (1) is neglected] that the evolution equation is given by $P(t) = -(1/\beta)f(t)$ and then v do admit an expansion in odd moments of $f(t)$.

Moreover, for small amplitudes ϵ_1 and ϵ_2, the leading term of the time-average velocity (8) reads

$$v = B\epsilon_1^2\epsilon_2 \cos(2\phi_1 - \phi_2 + \theta_0), \tag{9}$$

where $B = 3/(8M^3(\beta^2+\omega^2)\sqrt{\beta^2+4\omega^2})$ and $\theta_0 = -2\chi_1 + \chi_2$. This expression is in agreement with the prediction of our theory. Furthermore, in the limit $\beta \to 0$ we have $-2\chi_1 + \chi_2 \to \pi/2$, and in the combined limit $M \to 0$ and $\beta \to \infty$, with $\gamma =$ const., $-2\chi_1 + \chi_2 \to 0$. One can check that in the former case Eq. (1) is invariant under *time reversal* ($\mathcal{R} : t \mapsto -t$) provided $(\mathcal{R}f)(t) = f(-t) = f(t)$, and therefore $\theta_0 = \pi/2$ is the prediction of our theory. In the latter case, however, it is $(\mathcal{R}f)(t) = f(-t) = -f(t)$ that leaves Eq. (1) invariant and then our theory predicts $\theta_0 = 0$.

[1] O. H. Olsen and M. R. Samuelsen, Phys. Rev. B **28**, 210 (1983).
[2] M. Salerno and Y. Zolotaryuk, Phys. Rev. E **65**, 056603 (2002).