# Theoretical approach to two-dimensional traffic flow models

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In this paper we present a theoretical analysis of a recently proposed two-dimensional cellular automata model for traffic flow in cities with the ingredient of a turning capability. Numerical simulations of this model show that there is a transition between a freely moving phase with high velocity to a jammed state with low velocity. We study the dynamics of such a model, starting with the microscopic evolution equation, which will serve as a basis for further analysis. It is shown that a kinetic approach, based on the Boltzmann assumption, is able to provide a reasonably good description of the jamming transition. We further introduce a space-time continuous phenomenological model, leading to two partial differential equations whose preliminary results agree rather well with the numerical simulations.

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## I. INTRODUCTION

Traffic flow in networks is a problem of great technological interest. In systems where good communications are crucial (cities, computers, computer networks, etc.) the understanding of how undesirable, though possible, "traffic" jams form and evolve can lead to significant advances in design and planning of optimal strategies [1].

Traffic flow is a complex collective phenomenon that presents a rich phenomenology and, for the same reason, is difficult to deal with, both theoretically and numerically. In spite of that, given its great practical and technological impact, car traffic behavior on roads has been extensively studied. Fluid dynamics and kinetic theory have been traditionally used [2] as the theoretical tools to deal with the problem and treatment has been focused mainly on one-dimensional situations (highways with one or several lanes, with, maybe, some intersections, are the typical examples). Only recently [3] have models based on cellular automata (CA) been proposed to study this kind of problem. They present the advantage of being easier to handle numerically while catching the correct behavior of fluid-type systems. In recent years extensive studies on different one-dimensional CA models have been carried out, showing good agreement with previous theoretical and experimental results. These studies have also revealed new features of traffic flow. For example, the existence of self-organized criticality has been claimed in the behavior of car traffic on highways [4]. The situation is different in two dimensions (2D). The problem has not been treated in the fluid dynamics framework and CA models have opened the possibility to study it.

To our knowledge only two 2D models [5,6] have been reported. Both of them present a periodic city with two populations of cars moving in it. Reference [5] is a deterministic model where cars never turn, while the models in Ref. [6] are stochastic and cars are allowed to turn with certain probability. Extensive numerical simulations on these models [5–7] show the presence of a phase transition between a free moving (high velocity) uniform state and a jammed (low velocity) state, where the different populations of cars are separated. In the light of recent studies on CA models [8] the arising of this separation can be understood as a consequence of the violation of the semidetailed balance characteristic of traffic models.

At the moment there is no theoretical basis to explain all these results and it seems necessary to build one. The only exception has been a simple mean-field-like study [9] of the jamming transition of an extension of the model in Ref. [5]. In this paper we are going to perform a more ambitious theoretical analysis on which further work could be based. We will refer, as our working model, to the one called model $A$ in Ref. [6] (briefly described in Sec. II), but our procedure can easily be extended to other 2D CA models.

We will begin by writing down the microscopic evolution equations for a set of Boolean variables that describe exactly the dynamics of the system (Sec. III). These equations should be taken as the starting point for any further approximation. Using the microscopic variables we will be able to give an exact expression for the mean velocity of the system. We will show in Sec. IV, by means of very simple arguments, how the behavior of the mean velocity is correctly predicted in the freely moving phase. In Sec. V we will use the tools of kinetic theory to obtain an approximate macroscopic version of the microscopic equations by introducing an average over a nonequilibrium ensemble. We have started by considering the simplest approach: the Boltzmann approximation which assumes the breaking of spatial two-point correla-

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tions. Though simple, this approximation already shows the presence of a transition (by predicting the instability of the uniform phase) and the correct behavior of the velocity before and after the transition. It fails, however, in giving the density at which the transition occurs. Finally, in Sec. VI, we present a continuous (in space and time) description of the model. The purpose of this approach is to make the connection with the previous knowledge on traffic flow coming from fluid mechanics. In this paper a two-component fluid model has been presented to study 2D traffic flow. The continuous approximation will lead to a system of two coupled nonlinear partial differential equations that govern the evolution of the densities of the two types of cars present in the city. In fact they correspond to some limit of the Boltzmann equations of Sec. V. We will see how these equations also predict a phase transition as well as giving the correct behavior of the velocity of the two phases. We will end by writing our conclusions in Sec. VII.

## II. MODEL

As announced in the Introduction, in this paper we are going to deal with model $A$ of Ref. [6]. Briefly, this model can be described as follows. We have cars moving inside a town. The town is made of one-way perpendicular ($L$ horizontal and $L$ vertical) streets arranged in a square lattice with periodic boundary conditions. Vertical streets are oriented upwards and horizontal streets are oriented rightwards. There are $N$ cars moving inside this city. Cars are positioned at the crossings, and they can move to one of their nearest neighbors (allowed by the direction of the streets) at every time step. Two cars cannot be simultaneously at the same crossing. Among the cars, $N/2$ of them move rightward with probability $\gamma$ and upward with probability $1-\gamma$ (for symmetry reasons $0 \leq \gamma \leq 1/2$), and the remaining $N/2$ do it the other way around. Accordingly, half of the cars move mainly upward and half mainly rightward. Finally, there are traffic lights that permit vertical motion at even time steps and horizontal motion at odd time steps.

The dynamics of the model is as follows. At every time step the direction in which to move next is decided for every car according to its assigned probability. Then it is checked that the site where it is going to move is empty and that the motion is allowed by the traffic light; otherwise, the car will not be moved. Finally, all cars that can be moved are placed at their destination site and the next time step starts. We want to stress that the whole process is carried out simultaneously for all cars, which implies that, if a car leaves a site empty at a given time step $t$, it will not be occupied at the same time step. The fact that traffic lights allow motion alternately in vertical and horizontal streets prevents two cars from colliding at any crossing.

Notice finally that by setting $\gamma = 0$ we recover the deterministic model of Ref. [5].

The main results of this model are summarized in Fig. 1, which represents a plot of the average velocity ($v$) vs the density of cars ($n$) for different values of $\gamma$. The most striking feature of this figure is the first-order-like transition from a phase in which cars move freely to another in which cars are jammed. We refer to Ref. [6] for more details. In this reference it was shown that in the jammed phase cars arrange in diagonal strips—with the two types of cars roughly separated into two halves, and the question arose whether there were one or more

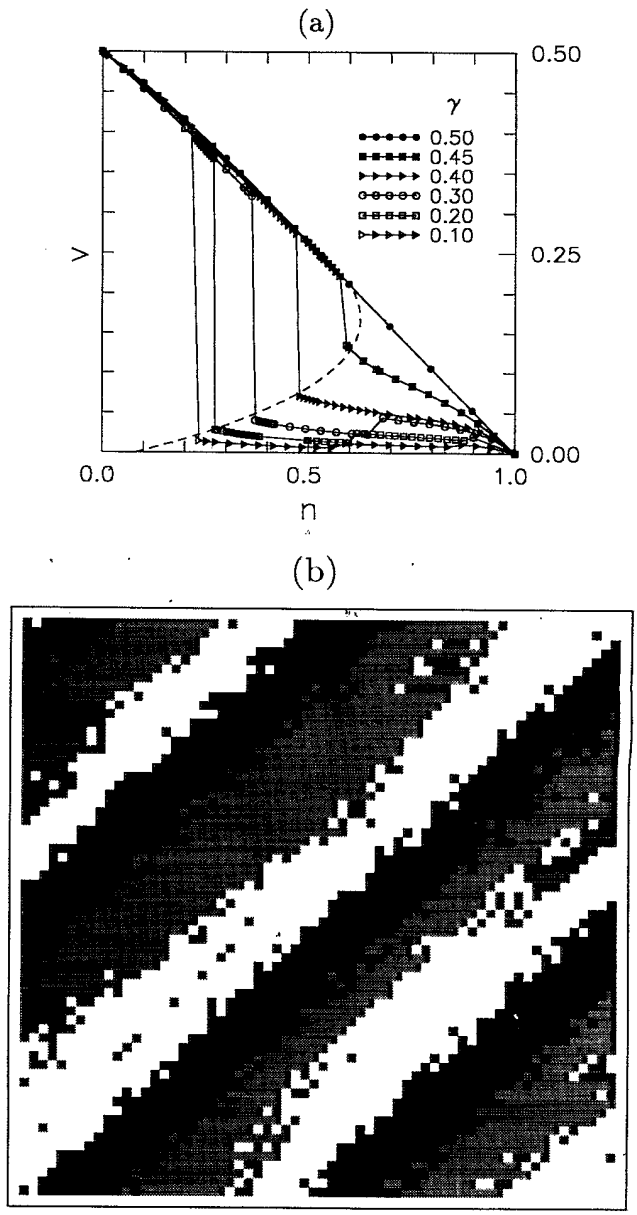


FIG. 1. Results of the simulations for the model of Ref. [6] for a city with $L = 64$: (a) phase diagram of the model, average velocity ($v$) vs car density ($n$), exhibiting the transitions from the freely moving phase to the jammed phase for a set of values of $\gamma$ (see [6] for details); (b) snapshot of a final run with $n = 0.672$ and $\gamma = 0.2$ (white and gray represent the two types of cars, while black represents empty sites). Notice the phase separation of the two populations of cars and the bandlike traffic jam.

strips. Arguments were given supporting the idea that, in very large systems, the jammed phase has a multistrip structure (in other words, the strips of this phase seem to have a "characteristic" size). In this paper we will try to reproduce analytically the phase diagram of Fig. 1 and to provide an answer to what the structure of the jammed phase is like.

## III. MICROSCOPIC EQUATIONS

The model can be described by a set of Boolean variables for every site of the lattice and every time step. This variables are the following:

(i) $\mu_{\mathbf{r}}^t$, the occupation number of site $\mathbf{r}$ at time $t$ by a car with probability $\gamma$ of moving horizontally (1 if occupied, 0 if not);

(ii) $\nu_{\mathbf{r}}^t$, the occupation number of site $\mathbf{r}$ at time $t$ by a car with probability $\gamma$ of moving vertically;

(iii) $\xi_{\mathbf{r}}^t$, a random variable which is 1 with probability $\gamma$ and 0 with probability $1-\gamma$;

(iv) $\eta_{\mathbf{r}}^t$, a random variable which is 0 with probability $\gamma$ and 1 with probability $1-\gamma$;

(v) $\sigma^t = t \bmod 2$, which is 1 if traffic lights permit horizontal movements and 0 if they permit vertical movements.

A given state of the system is completely characterized by specifying the whole set of occupation numbers, with the constraint $\mu_{\mathbf{r}}^t \nu_{\mathbf{r}}^t = 0$ (no two cars simultaneously at the same site). The random variables $\xi_{\mathbf{r}}^t$, $\eta_{\mathbf{r}}^t$, $\xi_{\mathbf{r}'}^{t'}$, and $\eta_{\mathbf{r}'}^{t'}$ are all independent two by two if $\mathbf{r} \neq \mathbf{r}'$ or $t \neq t'$; furthermore, they are also independent of the occupation numbers and of $\sigma^t$. These random variables are introduced to decide where cars are going to move to: $\xi_{\mathbf{r}}^t = 1$ ($\eta_{\mathbf{r}}^t = 1$) decides whether a $\mu$ ($\nu$) -type car placed at $\mathbf{r}$ at time $t$ is going to move horizontally and $\xi_{\mathbf{r}}^t = 0$ ($\eta_{\mathbf{r}}^t = 0$) decides whether it will move vertically.

With this set of Boolean variables, the evolution equations for the occupation numbers turn out to be

$$\begin{aligned}\mu_{\mathbf{r}}^{t+1} = \mu_{\mathbf{r}}^t \{ & \sigma^t \overline{\xi}_{\mathbf{r}}^t + \sigma^t \xi_{\mathbf{r}}^t (\mu_{\mathbf{r}+\mathbf{x}}^t + \nu_{\mathbf{r}+\mathbf{x}}^t) \\ & + \overline{\sigma}^t \xi_{\mathbf{r}}^t + \overline{\sigma}^t \overline{\xi}_{\mathbf{r}}^t (\mu_{\mathbf{r}+\mathbf{y}}^t + \nu_{\mathbf{r}+\mathbf{y}}^t)\} \\ & + \overline{\mu}_{\mathbf{r}}^t \overline{\nu}_{\mathbf{r}}^t \{ \sigma^t \xi_{\mathbf{r}-\mathbf{x}}^t \mu_{\mathbf{r}-\mathbf{x}}^t + \overline{\sigma}^t \overline{\xi}_{\mathbf{r}-\mathbf{y}}^t \mu_{\mathbf{r}-\mathbf{y}}^t \}, \end{aligned} \tag{1}$$

where we have introduced the short-hand notation $\overline{b} = 1 - b$ and where $\mathbf{x}$ and $\mathbf{y}$ are vectors representing the displacement of one site to the right and up, respectively. The evolution equation for $\nu_{\mathbf{r}}^t$ can be obtained from (1) by simply exchanging $\mu$ and $\nu$ and $\xi$ and $\eta$ (in what follows, whenever we write down an equation for the $\mu$-type cars, it must be understood that there is also another one for the $\nu$-type cars, which can be obtained by this or a similar replacement). In this way, the complete evolution of the system is specified by the above set of $2L^2$ Boolean equations plus a given initial state $\{\mu_{\mathbf{r}}^0, \nu_{\mathbf{r}}^0\}$ and periodic boundary conditions.

Equation (1) can be obtained from simple considerations on Boolean variables. First of all, each Boolean variable represents a condition that may or may not hold (for instance, $\mu_{\mathbf{r}}^t$ can be read as "if there is a $\mu$-type car at site $\mathbf{r}$ and time $t$...," and so on); secondly, the product of Boolean variables corresponds to the logical AND operation (whose result is always a new Boolean variable), and the addition of Boolean variables turns out to be the logical OR operation (whose result is another Boolean variable, provided all terms are mutually exclusive, i.e., of zero product). Now the purpose is to build a Boolean variable which is 1 if there is a $\mu$-type car at site $\mathbf{r}$ at time $t+1$ [Eq. (1)] and 0 if there is not. This is simply achieved by considering that (i) if there is a $\mu$-type car at site $\mathbf{r}$ and time $t$ it will still be there if it cannot move to the randomly chosen neighboring site ($\mathbf{r}+\mathbf{x}$ or $\mathbf{r}+\mathbf{y}$), either because it is occupied by any car or because the traffic light is red, and (ii) if there is no car at site $\mathbf{r}$ and time $t$ there will be one $\mu$-type car at the next time step if there is one of those cars at any of the neighboring sites ($\mathbf{r}-\mathbf{x}$ or $\mathbf{r}-\mathbf{y}$) willing to move to site $\mathbf{r}$ and its traffic light allows the movement. This provides all the terms of Eq. (1); for instance, $\mu_{\mathbf{r}}^t \sigma^t \overline{\xi}_{\mathbf{r}}^t$ is 1 only if there is a $\mu$-type car at site $\mathbf{r}$ at time $t$, the traffic lights allow horizontal movement, but it decides to move vertically; $\mu_{\mathbf{r}}^t \sigma^t \xi_{\mathbf{r}}^t (\mu_{\mathbf{r}+\mathbf{x}}^t + \nu_{\mathbf{r}+\mathbf{x}}^t)$ is 1 only if there is a $\mu$-type car at site $\mathbf{r}$ at time $t$, the traffic lights allow horizontal movement, it decides to move horizontally, but the site $\mathbf{r}+\mathbf{x}$ is occupied by any car, etc. The meaning of the remaining terms can be interpreted in the same way. It can be straightforwardly checked that all the terms are mutually exclusive (remember that $b\overline{b} = 0$) and that all possibilities have been considered.

Now, we can make use of the constraint $\mu_{\mathbf{r}}^t \nu_{\mathbf{r}}^t = 0$, which lets us write $\overline{\mu}_{\mathbf{r}}^t \overline{\nu}_{\mathbf{r}}^t = 1 - \mu_{\mathbf{r}}^t - \nu_{\mathbf{r}}^t$, to transform Eq. (1) into

$$\begin{aligned}\mu_{\mathbf{r}}^{t+1} = & (\sigma^t \overline{\xi}_{\mathbf{r}}^t + \overline{\sigma}^t \xi_{\mathbf{r}}^t) \mu_{\mathbf{r}}^t + \sigma^t \xi_{\mathbf{r}-\mathbf{x}}^t \mu_{\mathbf{r}-\mathbf{x}}^t + \overline{\sigma}^t \overline{\xi}_{\mathbf{r}-\mathbf{y}}^t \mu_{\mathbf{r}-\mathbf{y}}^t \\ & + \sigma^t \xi_{\mathbf{r}}^t \mu_{\mathbf{r}}^t (\mu_{\mathbf{r}+\mathbf{x}}^t + \nu_{\mathbf{r}+\mathbf{x}}^t) - \sigma^t \xi_{\mathbf{r}-\mathbf{x}}^t \mu_{\mathbf{r}-\mathbf{x}}^t (\mu_{\mathbf{r}}^t + \nu_{\mathbf{r}}^t) \\ & + \overline{\sigma}^t \overline{\xi}_{\mathbf{r}}^t \mu_{\mathbf{r}}^t (\mu_{\mathbf{r}+\mathbf{y}}^t + \nu_{\mathbf{r}+\mathbf{y}}^t) - \overline{\sigma}^t \overline{\xi}_{\mathbf{r}-\mathbf{y}}^t \mu_{\mathbf{r}-\mathbf{y}}^t (\mu_{\mathbf{r}}^t + \nu_{\mathbf{r}}^t). \end{aligned} \tag{2}$$

In this form it can be clearly seen that the highest order terms contributing to the evolution equations are quadratic in the occupation numbers, something whose relevance will be made clear in the next section.

The initial configuration must be such that it satisfies the requirements $\sum_{\mathbf{r}} \mu_{\mathbf{r}}^0 = \sum_{\mathbf{r}} \nu_{\mathbf{r}}^0 = N/2$, and $\mu_{\mathbf{r}}^0 \nu_{\mathbf{r}}^0 = 0$; the former counts the number of cars of each type whereas the latter express the impossibility of having two cars at the same site. It is straightforward to check that

$$\sum_{\mathbf{r}} \mu_{\mathbf{r}}^t = \sum_{\mathbf{r}} \nu_{\mathbf{r}}^t = \frac{N}{2} \tag{3}$$

are constants of motion for the evolution equations (2) and that $\mu_{\mathbf{r}}^t \nu_{\mathbf{r}}^t = 0$ is maintained in the evolution.

We need now a microscopic definition for the *average* velocity in order to be able to obtain what could be called the "equation of state" (average velocity vs density of cars) for this model. If we define the *mean* velocity at time $t$ as the number of moved cars divided by $N$, this can be expressed in terms of the occupation numbers:

$$v(t) = \frac{1}{2N} \sum_{\mathbf{r}} \left\{ \left(\mu_{\mathbf{r}}^{t+1} - \mu_{\mathbf{r}}^{t}\right)^2 + \left(\nu_{\mathbf{r}}^{t+1} - \nu_{\mathbf{r}}^{t}\right)^2 \right\} . \quad (4)$$

This equation can be easily visualized if one realizes that $\mu_{\mathbf{r}}^{t+1} - \mu_{\mathbf{r}}^{t}$ is zero only if there is no $\mu$-type car at site $\mathbf{r}$ at times $t$ and $t+1$ or if there is one that remains there at both times; in other words, if there is no movement of any $\mu$-type car at site $\mathbf{r}$ between $t$ and $t+1$. Otherwise the value is 1 (if the car arrives at time $t+1$) or $-1$ (if the car leaves at time $t+1$). The square removes the sign and the factor 1/2 lets the expression count only the cars that arrive and not those which leave. The same applies to the term related to $\nu$-type cars. Notice that only one of the two terms may be 1.

The average velocity was computed in Ref. [6] as the limit

$$v = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} v(t) , \quad (5)$$

which, if the system reaches a steady state, corresponds to its mean velocity [10].

Before we go any further we would like to remark that all the formalism introduced in this section also applies to the deterministic model [5] by simply setting $\gamma = 0$, although in what follows we will make explicit use of the randomness to obtain information out of the evolution equations.

## IV. LOW DENSITY LIMIT

Figure 1 shows that the behavior of the $v$-$n$ curves in the free moving phase is, to a high accuracy, $\gamma$ independent (for $\gamma > 0$). In fact, they decrease almost linearly with slope $-1/2$. Reproducing analytically this remarkable property is a first important result which can be derived from the microscopic equations. It will be the main goal of this section. To this purpose notice that Eq. (4) can be rewritten

$$v(t) = 1 - \frac{1}{n} \left( \left[\mu^{t+1}\mu^{t}\right] + \left[\nu^{t+1}\nu^{t}\right] \right) , \quad (6)$$

where the following notation has been introduced: $[A^t] \equiv L^{-2} \sum_{\mathbf{r}} A_{\mathbf{r}}^t$, the average over the whole lattice of a given function $A_{\mathbf{r}}^t$, and the car density $n$, defined as $n \equiv NL^{-2}$. In getting (6) it has been used that any Boolean variable $b$ satisfies $b^2 = b$ and that $[\mu^t] = [\nu^t] = n/2$. Time correlations can be transformed into space correlations by introducing the dynamics, i.e., by using the equations of motion (2):

$$\left[\mu^{t+1}\mu^{t}\right] = \sigma^t \left\{ \left[\overline{\xi}^t \mu^t\right] + \left[\xi^t \mu^t \mu_{\mathbf{x}}^t\right] + \left[\xi^t \mu^t \nu_{\mathbf{x}}^t\right] \right\} + \overline{\sigma}^t \left\{ \left[\xi^t \mu^t\right] + \left[\overline{\xi}^t \mu^t \mu_{\mathbf{y}}^t\right] + \left[\overline{\xi}^t \mu^t \nu_{\mathbf{y}}^t\right] \right\} , \quad (7)$$

and the counterpart for the $\nu$'s (obtained by exchanging $\xi$ and $\eta$ and $\mu$ and $\nu$ in the equation above). We have denoted with the subscripts $\mathbf{x}$ or $\mathbf{y}$ a translation by those unit vectors. Equation (7) simplifies in the $L \to \infty$ limit, for then the random variables can be replaced by their averages ($\gamma$ or $1-\gamma$), as they are independent of the car variables. Furthermore, in the low density limit ($n \to 0$) we can assume that the occupation numbers become independent variables in the uniform steady state ($[\mu^t \mu_{\mathbf{x}}^t] \sim [\mu^t][\mu_{\mathbf{x}}^t] = n^2/4$), and similarly for the other seven spatial correlations—those in (7) and its counterpart. In principle (though we will later come back to this point again), this hypothesis is justified by the fact that, due to the random dynamics, at very low densities cars have enough time to "forget" between two successive encounters. Therefore, in this regime,

$$\left[\mu^{t+1}\mu^{t}\right] \sim \sigma^t \left\{ \overline{\gamma}\frac{n}{2} + 2\gamma \left(\frac{n}{2}\right)^2 \right\} + \overline{\sigma}^t \left\{ \gamma\frac{n}{2} + 2\overline{\gamma} \left(\frac{n}{2}\right)^2 \right\}. \quad (8)$$

Inserting this expression into the equation for the mean velocity (6) we get

$$v(t) \sim \tfrac{1}{2}(1-n) , \quad (9)$$

and since this formula corresponds to the uniform steady state it is the average velocity (5). This expression for $v$ is $\gamma$ independent, which is in good agreement with the results obtained in the simulations in the freely moving phase; in particular it predicts the slope $-1/2$ of the $v$-$n$ curves for small $n$.

As Eq. (9) is $\gamma$ independent, in Ref. [6] we used this argument to justify that the slope is $-1/2$ for *all* values of $\gamma$, even for $\gamma = 0$. The results of Ref. [5] do not agree with this because they show clearly that $v = 1$ for the freely moving phase and $v = 0$ for the jammed phase. To settle this matter we have done a more careful analysis and have found that a connection between both results can be made, explaining the differences as a consequence of the boundary conditions [11]. This analysis is out of the scope of this paper and is available elsewhere [12].

## V. BOLTZMANN APPROXIMATION

Equation (2) and its $\nu$ counterpart make it feasible to employ the methods of the standard kinetic theory [13,14] in its lattice-gas version [15,16] in order to obtain macroscopic properties of the system. To this purpose let us introduce a nonequilibrium ensemble in which all initial conditions $\{\mu_{\mathbf{r}}^0, \nu_{\mathbf{r}}^0\}$ are equally weighted. We will denote the average over this ensemble by $\langle\cdot\rangle$. When we average over the Boolean variables $\mu_{\mathbf{r}}^t$ and $\nu_{\mathbf{r}}^t$, we obtain real variables—$u(\mathbf{r},t)$ and $w(\mathbf{r},t)$, respectively—ranging from 0 to 1 which measure the average number of initial configurations for which the site $\mathbf{r}$ is occupied at time $t$ by the corresponding type of car (henceforth, the average occupation). Now, if we average the microscopic evolution equations (2) over this ensemble we obtain

$$
\begin{aligned}
u(\mathbf{r},t+1) = {} & (\sigma^t\overline{\gamma}+\overline{\sigma}^t\gamma)u(\mathbf{r},t)+\sigma^t\gamma u(\mathbf{r}-\mathbf{x},t)+\overline{\sigma}^t\overline{\gamma}u(\mathbf{r}-\mathbf{y},t) \\
& +\sigma^t\gamma\left(\langle\mu^t_{\mathbf{r}}\mu^t_{\mathbf{r}+\mathbf{x}}\rangle-\langle\mu^t_{\mathbf{r}-\mathbf{x}}\mu^t_{\mathbf{r}}\rangle+\langle\mu^t_{\mathbf{r}}\nu^t_{\mathbf{r}+\mathbf{x}}\rangle-\langle\mu^t_{\mathbf{r}-\mathbf{x}}\nu^t_{\mathbf{r}}\rangle\right) \\
& +\overline{\sigma}^t\overline{\gamma}\left(\langle\mu^t_{\mathbf{r}}\mu^t_{\mathbf{r}+\mathbf{y}}\rangle-\langle\mu^t_{\mathbf{r}-\mathbf{y}}\mu^t_{\mathbf{r}}\rangle+\langle\mu^t_{\mathbf{r}}\nu^t_{\mathbf{r}+\mathbf{y}}\rangle-\langle\mu^t_{\mathbf{r}-\mathbf{y}}\nu^t_{\mathbf{r}}\rangle\right). \qquad (10)
\end{aligned}
$$

In writing Eq. (10) we have used the fact that $\xi^t_{\mathbf{r}}$ and $\eta^t_{\mathbf{r}}$ and any combination of occupation numbers are uncorrelated at the same time step. These equations are not closed due to the presence of the quadratic terms which, on the other hand, contain the interaction of the model and therefore cannot be ignored. Evolution equations for the averages of these quadratic terms can be written, but cubic and quartic terms will appear, and so on, giving rise to a hierarchy similar to the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) one [14]. Being able to write down a closed set of equations for the average occupations requires an approximation. The simplest one is to assume that average occupations at different sites are *always* uncorrelated. In kinetic theory this approximation is known as the Boltzmann approximation (or molecular chaos hypothesis). For our purpose, this approximation amounts to writing

$$\langle\mu^t_{\mathbf{r}}\mu^t_{\mathbf{r}'}\rangle = u(\mathbf{r},t)u(\mathbf{r}',t)\,, \qquad (11a)$$

$$\langle\mu^t_{\mathbf{r}}\nu^t_{\mathbf{r}'}\rangle = u(\mathbf{r},t)w(\mathbf{r}',t)\,, \qquad (11b)$$

$$\langle\nu^t_{\mathbf{r}}\nu^t_{\mathbf{r}'}\rangle = w(\mathbf{r},t)w(\mathbf{r}',t)\,, \qquad (11c)$$

for any pair of different sites $\mathbf{r}$ and $\mathbf{r}'$ and for every time step $t$. Physically speaking, the molecular chaos hypothesis assumes that two colliding "particles" (cars in our model) have never met before, or if they have, all their mutual influence has been lost. This hypothesis is expected to be true in the limits of short times (cars have never met before) or vanishing densities (the probability of an encounter is small enough so as to lose all information between successive encounters). We have already made use of this fact in Sec. IV.

A further simplification may be introduced; the replacement of the traffic light variable $\sigma^t$ by its time average 1/2. This approximation is expected to hold in the macroscopic regime (which we are interested in), where the microscopic details are not seen. However, the study of microscopic scales would require one to keep $\sigma^t$ to its original values 0,1 alternately. We will go back to this point later.

In summary, the ensemble average together with these two approximations leads to a closed set of equations (henceforth referred to as the *Boltzmann equations*) for the averaged variables $u(\mathbf{r},t)$ and $w(\mathbf{r},t)$:

$$
\begin{aligned}
u(\mathbf{r},t+1) = {} & \frac{1}{2}u(\mathbf{r},t)+\frac{\gamma}{2}u(\mathbf{r}-\mathbf{x},t)+\frac{\overline{\gamma}}{2}u(\mathbf{r}-\mathbf{y},t) \\
& +\frac{\gamma}{2}u(\mathbf{r},t)\{u(\mathbf{r}+\mathbf{x},t)+w(\mathbf{r}+\mathbf{x},t)\}-\frac{\gamma}{2}u(\mathbf{r}-\mathbf{x},t)\{u(\mathbf{r},t)+w(\mathbf{r},t)\} \\
& +\frac{\overline{\gamma}}{2}u(\mathbf{r},t)\{u(\mathbf{r}+\mathbf{y},t)+w(\mathbf{r}+\mathbf{y},t)\}-\frac{\overline{\gamma}}{2}u(\mathbf{r}-\mathbf{y},t)\{u(\mathbf{r},t)+w(\mathbf{r},t)\}. \qquad (12)
\end{aligned}
$$

Iteration of the Boltzmann equations will give a mean-field-like evolution of our model. This evolution will lack some of the effects caused by correlations, but it still keeps many of the interesting features of the model. Furthermore, the mean-field description allows us to obtain analytically many physical quantities of relevant interest. The remainder of this section is devoted to studying the Boltzmann equations. Section V A deals with the linear regime where we determine the stability of the homogeneous solution, whereas in Sec. V B we present the results of a numerical simulation performed directly on the Boltzmann equations and compare them with the linear regime as well as with the simulations of the full model [6] obtained from the exact microdynamics, Eqs. (2).

### A. Linear stability analysis

The Boltzmann equations (12) have, for any value of the turning parameter $\gamma$, the uniform solution

$$u(\mathbf{r},t)=w(\mathbf{r},t)=n/2\,,\quad \forall t\geq 0\,. \qquad (13)$$

This is what simulations show for small enough density [6]. However, above a threshold (depending on $\gamma$) the simulations show a steady nonuniform pattern, which means that the uniform solution becomes unstable against small perturbations. Thus, settling the stability of (13) requires studying the evolution of small perturbations about it, i.e., we need to look for solutions of the form

$$u(\mathbf{r},t)=\frac{n}{2}+\delta u(\mathbf{r},t)\,,\quad w(\mathbf{r},t)=\frac{n}{2}+\delta w(\mathbf{r},t)\,, \qquad (14)$$

which introduced into (12) give, to linear order

$$
\begin{aligned}
\delta u(\mathbf{r},t+1) = {} & \frac{1}{2}\Bigg\{\left(1+\frac{n}{2}\right)\delta u(\mathbf{r},t)+\gamma\frac{n}{2}\delta u(\mathbf{r}+\mathbf{x},t) \\
& +\gamma(1-n)\delta u(\mathbf{r}-\mathbf{x},t)+\overline{\gamma}\frac{n}{2}\delta u(\mathbf{r}+\mathbf{y},t) \\
& +\overline{\gamma}(1-n)\delta u(\mathbf{r}-\mathbf{y},t)\Bigg\}-\frac{n}{4}\{\delta w(\mathbf{r},t) \\
& -\gamma\delta w(\mathbf{r}+\mathbf{x},t)-\overline{\gamma}\delta w(\mathbf{r}+\mathbf{y},t)\}. \qquad (15)
\end{aligned}
$$

Equation (15) is linear but nonlocal, because it involves the average car occupations at five nodes ($\mathbf{r}$ and its four nearest neighbors). By taking the discrete Fourier transform, defined as

$$\delta\widehat{f}(\mathbf{k},t) = \sum_{\mathbf{r}} e^{-i\mathbf{k}\cdot\mathbf{r}}\delta f(\mathbf{r},t) \tag{16}$$

(where $k_{x(y)} = 2\pi q_{x(y)}/L$, $q_{x(y)} = 0,1,2,\ldots,L-1$, and $f$ is either $\mu$ or $\nu$) Eq. (15) becomes

$$\begin{pmatrix} \delta\widehat{u}(\mathbf{k},t+1) \\ \delta\widehat{w}(\mathbf{k},t+1) \end{pmatrix} = \Omega(\mathbf{k}) \begin{pmatrix} \delta\widehat{u}(\mathbf{k},t) \\ \delta\widehat{w}(\mathbf{k},t) \end{pmatrix}, \tag{17}$$

where $\Omega(\mathbf{k})$ is the $\mathbf{k}$-dependent $2\times2$ linear evolution operator whose elements are given by

$$\Omega_{11} = \frac{1}{2} + \frac{n}{4}\{1+\gamma S_x + \overline{\gamma} S_y\} + \frac{1-n}{2}\{\gamma S_x^* + \overline{\gamma} S_y^*\}, \tag{18a}$$

$$\Omega_{12} = \frac{n}{4}\{\gamma S_x + \overline{\gamma} S_y - 1\}, \tag{18b}$$

with $S_{x(y)} \equiv e^{i\mathbf{k}_{x(y)}}$. The elements $\Omega_{21}$ and $\Omega_{22}$ are obtained from Eqs. (18) by exchanging $\gamma$ and $\overline{\gamma}$ in $\Omega_{12}$ and $\Omega_{11}$, respectively. Iteration of Eq. (17) yields the dynamics of the system at the linear level in the Boltzmann approximation. The information on the stability of the uniform state is contained in the eigenvalues of the linear evolution operator $\Omega(\mathbf{k})$, written for convenience as

$$\Omega(\mathbf{k})\psi_j(\mathbf{k}) = e^{z_j(\mathbf{k})}\psi_j(\mathbf{k}), \qquad j = 1,2. \tag{19}$$

Hence the dynamics at the linear level is simply expressed as

$$\begin{pmatrix} \delta\widehat{u}(\mathbf{k},t) \\ \delta\widehat{w}(\mathbf{k},t) \end{pmatrix} = U(\mathbf{k}) \begin{pmatrix} e^{z_1(\mathbf{k})t} & 0 \\ 0 & e^{z_2(\mathbf{k})t} \end{pmatrix} \times U^{-1}(\mathbf{k}) \begin{pmatrix} \delta\widehat{u}(\mathbf{k},0) \\ \delta\widehat{w}(\mathbf{k},0) \end{pmatrix}, \tag{20}$$

where the first (second) column of $U(\mathbf{k})$ is the eigenvector $\psi_{1(2)}(\mathbf{k})$. Note here that the two eigenvalues must satisfy

$$\lim_{\mathbf{k}\to\mathbf{0}} z_j(\mathbf{k}) = 0 \tag{21}$$

as a consequence of the conservation laws (3) [for the sake of simplicity we will be referring to $z_j(\mathbf{k})$ as the eigenvalues, though they are actually their logarithm].

Whenever $\mathrm{Re}z_j(\mathbf{k}) < 0$ ($j = 1,2$) for all $\mathbf{k}$, the amplitude of all eigenmodes, $\psi_j(\mathbf{k})$, will vanish exponentially with time and finally disappear, recovering the uniform state. However, as soon as a value of $\mathbf{k}$ appears such that $\mathrm{Re}z_{j_m}(\mathbf{k}) > 0$ (for $j_m = 1$ or 2 or both), then the amplitude of the corresponding eigenmode $\psi_{j_m}(\mathbf{k})$ will grow exponentially with time; in other words, the uniform state will become unstable, and inhomogeneous spatial structure will develop. Therefore the spectrum of the linear evolution operator determines the macroscopic behavior of the system. There is more information that can be extracted from the spectrum when the uniform state is unstable [17]. In general, there will be a wave vector $\mathbf{k}_m$ for which $\mathrm{Re}z_{j_m}(\mathbf{k}_m)$ (for some $j_m = 1,2$) is maximum. Fluctuations with this wave vector will dominate as $t \to \infty$ and consequently the structure will possess a typical length scale of $\lambda_m \equiv 2\pi/k_m$, and will be formed along the direction of $\mathbf{k}_m$. Furthermore, the onset time of the instability is given by $[\mathrm{Re}z_{j_m}(\mathbf{k}_m)]^{-1}$ [see Eq. (20)].

The eigenvalues $z_j(\mathbf{k})$ are easily obtained from the elements of the linear evolution operator, given in (18). A typical plot of $\mathrm{Re}z_{j_m}(\mathbf{k})$ is shown in Fig. 2 for $n = 0.8$ and $\gamma = 0.1$. According to the numerical simulations (Fig. 1) for this parameter set the system is jammed, i.e., the system is nonuniform since cars cluster in bands which form an angle $\theta = \pi/4$ with the positive $x$ axis. From Fig. 2 we see that the linear stability analysis agrees with this fact as there exist regions where $\mathrm{Re}z_{j_m}(\mathbf{k})$ is positive with maxima located on a line forming an angle of $\theta = 3\pi/4$ with the positive $x$ axis, i.e., perpendicular to the bands of the system. From the location of the maxima we can obtain the typical size of the bands or, equivalently, the number of bands which would appear in a finite size simulation. We will give more details about this point later in this section. We have not shown the second eigenvalue, because it satisfies $\mathrm{Re}z(\mathbf{k}) < 0$ for all values of $\mathbf{k}$, so it does not affect the previous discussion.

The traffic jams of this model are always bands like those of Fig. 1(b) (i.e., with $\theta = \pi/4$). Hence the maxima of $\mathrm{Re}z_{i_m}(\mathbf{k})$ appear along the direction given by $\widehat{\mathbf{k}} = (-1/\sqrt{2}, 1/\sqrt{2})$. Therefore, in order to look for these maxima in the expressions for $z_i(\mathbf{k})$, it is enough to choose a wave vector of the form $\mathbf{k} = k\widehat{\mathbf{k}}$. This means that $S_x = S_y^* = \exp(-ik/\sqrt{2})$. For this particular direction the eigenvalues take a simple form:

$$e^{z_\pm(k)} = 1 - s^2\left(1-\frac{n}{2}\right) \pm s\left\{\frac{s^2n^2}{4} - 2(1-n) \times \left(\frac{1}{2}-n\right)\zeta^2(1-s^2)\right\}^{1/2}, \tag{22}$$

where $s \equiv \sin(k/2\sqrt{2})$ and $\zeta \equiv 1-2\gamma$. It is straight-

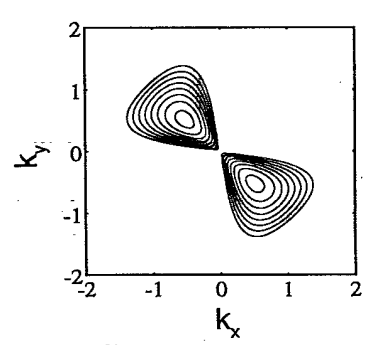


FIG. 2. Plot of the real part of the largest eigenvalue of the linear Boltzmann operator (only positive parts are seen). The set of parameters is $n = 0.8$ and $\gamma = 0.1$. Regions where $\mathrm{Re}z(\mathbf{k})$ is positive appear along the direction forming an angle $\theta = 3\pi/4$ with the $x$ axis (see text).

forward to check that $\mathrm{Re}z_+(k) > 1$ for some $k$'s if and only if $n > 1/2$. In consequence, this analysis predicts a jamming transition for $n > 1/2$ irrespective of the value of $\gamma$. The existence of this transition is the most important result of this section; furthermore, the prediction for the transition density is reasonably good for $\gamma$ close to 1/2, the highest randomness [see Fig. 1(a)]. However, it disagrees with the simulations for smaller $\gamma$'s.

A comment about the role of the traffic light variables $\sigma^t$ is needed here. Figure 2 (and the discussion about it) has been obtained for $\sigma^t = 1/2$. Keeping the variables $\sigma^t$ to their original values, 0,1, alternately, one obtains two evolution operators, one for even time steps, $\Omega^e(\mathbf{k})$, and another one for odd time steps, $\Omega^o(\mathbf{k})$. Then the eigenvalue problem in Eq. (19) has to be formulated in terms of an effective evolution operator given by $\{\Omega^e(\mathbf{k})\Omega^o(\mathbf{k})\}^{1/2}$. However, in doing so, one has to face the fact that $\Omega^o(\mathbf{k})$ and $\Omega^e(\mathbf{k})$ do not commute. Thus we end up with three sets of eigenvalues, those of Eq. (19), of $\{\Omega^e(\mathbf{k})\Omega^o(\mathbf{k})\}^{1/2}$, and of $\{\Omega^o(\mathbf{k})\Omega^e(\mathbf{k})\}^{1/2}$. The largest difference between these three sets is for the wave vectors $\mathbf{k}$ located at the edges of the Brillouin zone ($k \simeq \pi$), related to the microscopic behavior when the system is observed at scales of the order of a few lattice spacings. In contrast, for wave vectors at the center of the Brillouin zone ($k \simeq 0$) the three sets coincide. This is the region we are interested in, because it describes the macroscopic behavior. As a summary, the macroscopic behavior is not affected by the traffic lights being 0,1 or its average 1/2. For this reason, and for the sake of simplicity, we have made all the calculations with $\sigma^t$ replaced by 1/2.

According to kinetic theory, for each quantity conserved by the dynamics there exists an eigenvector as given by Eq. (19) whose eigenvalue satisfies the relation (21). In the long-wavelength (small-wave-number) limit, the eigenvalues can be written as

$$z_j(\mathbf{k}) = ikc_j + (ik)^2 D_j + \cdots \qquad (\mathbf{k} \to \mathbf{0}). \tag{23}$$

The quantity $c_j$ is interpreted as the speed of propagation of the mode, while $D_j$ is the corresponding diffusivity. In the case of fluid dynamics in three dimensions, there are five conserved quantities, and five of such modes: two sound modes, two shear modes, and finally a thermal mode. This is a general result of kinetic theory and also holds for lattice gases as models for thermal fluids [18]. In the case of a model of concentration diffusion leading to the Fick law, there is only one mode (the concentration); $c_j$ is then the drift velocity and $D_j$ the diffusion coefficient. The two eigenvalues of our model [Eq. (22)] correspond to the two conservation laws (3), so that expanding them in powers of the wave vector $k$, as in Eq. (23), we can identify the two (opposite) speeds of propagation (that are imaginary for $n < \frac{1}{2}$) [19] and the (unique) diffusion coefficient, with the result (for $n < 1/2$)

$$c_\pm = \pm\tfrac{1}{2}\zeta(1-n)^{1/2}\left(n-\tfrac{1}{2}\right)^{1/2}, \tag{24a}$$

$$D = \tfrac{1}{16}\left[2-n-(1-n)(1-2n)\zeta^2\right]. \tag{24b}$$

We have to state here that this velocity has nothing to do with the remnant velocity measured from the simulations, but it is the velocity with which long-wavelength fluctuations propagate (the so-called kinematic velocity in fluid dynamics studies of traffic flow [2]).

As was said before, by calculating the maximum with respect to $k$ of $\mathrm{Re}z_+(\mathbf{k})$, with $\mathbf{k}$ along the direction of $\theta = 3\pi/4$ [Eq. (22)] we obtain the characteristic length of the system as $\lambda_m \equiv 2\pi/k_m$. The relevance of this quantity is that it provides information about the typical center-to-center band separation in the jammed phase and consequently determines the number of bands. We will not write here the expression for $\lambda_m$, because it is very cumbersome. Instead, in Fig. 3 we plot the value of $\lambda_m$ as a function of the density $n$ for some values of the randomness $\gamma$. It can be seen that, apart from both edges ($n \gtrsim 1/2$ and $n \lesssim 1$), the bands have a typical width of a few tens of lattice spacings, in qualitative agreement with the simulations. However, in actual simulations the characteristic length may not be the natural one (that the system would have if it were infinite) because of a competition with the finite size of the system. If the system is not large enough to form an integer number of bands of width $\lambda_m$ it will form fewer bands, but concentrating all the cars in the existing bands. Therefore the characteristic length will be modified. However, the value of $\lambda_m$ we have obtained can still give a good prediction of the most stable number of bands in the simulations at finite $L$. The number of bands predicted by the Boltzmann theory is simply $L/\lambda_m$. If $L/\lambda_m$ is not an integer, the system will have the trend to form $\lfloor L/\lambda_m \rfloor$ bands (where $\lfloor x \rfloor$ means the largest integer smaller than or equal to $x$). Reading off Fig. 3 for the case $\gamma = 0.4$ we obtain $\lfloor L/\lambda_m \rfloor = 1$ for all $n > 1/2$, in perfect agreement with what is shown in Fig. 1(a) (made for a city with $L = 64$). The absence of jumps in this curve (in the jammed phase) means that only one band has been formed. For the case $\gamma = 0.3$, Fig. 3 predicts $\lfloor L/\lambda_m \rfloor = 1$ for $n \lesssim 0.6$, $\lfloor L/\lambda_m \rfloor = 2$ for $0.6 \lesssim n \lesssim 0.7$, and $\lfloor L/\lambda_m \rfloor = 3$ for $0.7 \lesssim n \lesssim 0.95$. Inspection of Fig. 1 shows that there is only one band for $n \lesssim 0.65$, while for $n \gtrsim 0.65$ there are two bands, as

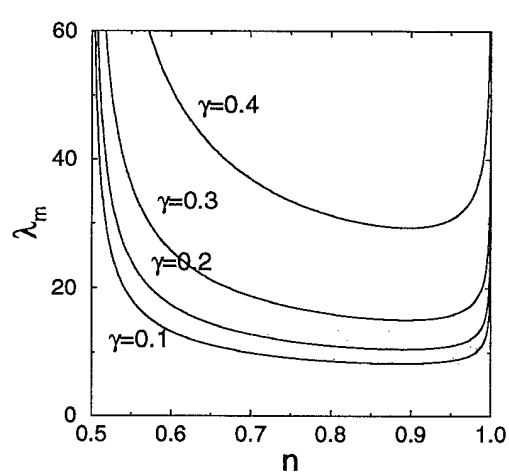


FIG. 3. Typical band-to-band distance $\lambda_m$ in the jammed phase vs the density $n$ for several values of the randomness $\gamma$.

indicated from the increase of the velocity. For higher densities, although it cannot be seen in Fig. 1, occasionally some simulations end up in a three-band structure [6]. Unfortunately, the agreement at this level between the Boltzmann predictions and the simulations is poorer the smaller the randomness. For $\gamma = 0.2$ and $\gamma = 0.1$ the number of bands given by the Boltzmann theory for a system size of $L = 64$ at intermediate densities is about five or six, while in actual simulations such number is smaller (one, two, or three bands, but never more). This result could be explained by the fact that the higher the number of bands the more difficult to see them in simulations, because of band coalescence at their early stages of formation. On the other hand, the Boltzmann approximation was shown to be worse the lower $\gamma$, because it predicted the jamming transition for $n = 1/2$, a result that can be valid for $\gamma$ close to 1/2 (see Fig. 1), but certainly is not for small $\gamma$, where experimentally one finds that $n$ can be as low as $n \simeq 0.24$.

At both edges of the plot presented in Fig. 3 the quantity $\lambda_m$ goes to infinity as $\lambda_m \sim (n-1/2)^{-1/2}$ for $n \gtrsim 0.5$ while $\lambda_m \sim (1-n)^{-1/4}$ when $n \lesssim 1$. The former suggests that just after the transition only one band is formed; the latter means that all bands coalesce into one which fills up the whole system. As $\lambda_m \to \infty$, $k_m \to 0$; thus, as the eigenvalues obey the restriction (21), the onset time of the instability (given by $[\mathrm{Re}z(\mathbf{k}_m)]^{-1}$) diverges just above the transition. This is indeed observed in the numerical simulations, where extremely long relaxation times are needed near the jamming transition. For $\gamma$'s close to 1/2 the onset time of the instability also diverges [see Eq. (22)]. For intermediate $n$ (in the jammed phase) the typical onset times range from a few hundreds to a few thousands of time steps.

The last remark of this subsection concerns the special case $\gamma = 1/2$. For this value, for which there is no distinction between cars of type $\mu$ and $\nu$, Eq. (22) predicts negative eigenvalues for any value of the density. As a consequence, there is no jamming transition for this particular case, in agreement with the numerical simulations [Fig. 1(a)].

### B. Simulation of the Boltzmann equations

All the information presented in the previous subsection was obtained within the Boltzmann approximation, but in the linear regime. This regime is only valid for small perturbations around the homogeneous state. However, when the perturbations are no longer small, as happens once the jammed phase appears, the full nonlinear Boltzmann equation takes over, and the information given by the linear theory may not hold [20]. In particular, no linear theory can give the saturation (i.e., completely filled nodes) occurring in the bands in the jammed phase.

In order to study the jammed phase we have performed simulations of the full Boltzmann equations (12), starting with a uniform state slightly perturbed at random. The evolution equations (12) are then iterated until a stationary state is reached. This final state can be either again uniform (when it is stable) or nonuniform, exhibiting a traffic jam. This latter situation is illustrated by Fig. 4 for $n = 0.6$ and $\gamma = 0.2$, where a plot of the average occupation profiles in the steady state is shown on the line perpendicular to the bands. We clearly see the band structure, with a completely saturated region ($u = 1$ or $w = 1$), bounded by partly filled, very narrow layers. The result is similar to that obtained in direct simulations on the microscopic model [6]. The center-to-center band distance as measured from the simulations is equal to 17 (in lattice units), while the linear stability analysis predicts $\lambda_m \simeq 16$, in very good agreement.

The phase transition is numerically obtained for the same region of parameters as those predicted by the linear theory ($n = 1/2$ for all $\gamma$). This result supports the idea that the fact that the transition is $\gamma$ independent is an effect of the Boltzmann approximation itself, and not of the linearization. In fact, from the general theory of stability of dynamical systems, one can prove that the stability is given by the linearized equations (unless zero eigenvalues are obtained), and is not modified by the nonlinearities. Therefore it is the breakup of the correlations that gives the degeneracy in the transition density. So finding a $\gamma$-dependent transition requires going beyond the Boltzmann approximation. Some work along this line is currently being done.

From the simulation of the full Boltzmann equations we can obtain a quantitative prediction for the velocity in the jammed phase. In order to measure the velocity we cannot use the microscopic definition given in Eq. (4), because in the steady state the average occupations do not change in time. However, we can use the modified Eq. (7), where no time variations appear, to compute the velocity. Figure 5 summarizes these results. In spite of the $\gamma$-independent transition, there is a strong similarity between this figure and the phase diagram obtained from the simulations of the microscopic model (Fig. 1); in particular, the slope $-1/2$ before the transition is properly

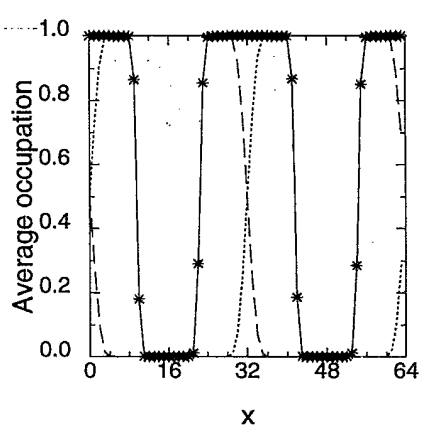


FIG. 4. Jammed phase for the set of parameters $n = 0.6$, $\gamma = 0.2$, as obtained from the simulation of the full Boltzmann equations (12) on a lattice of size 64×64. We only plot here the occupation profiles along the line perpendicular to the band. Cars of type $\mu$ ($\nu$) are represented by a dotted (dashed) line. The total population is plotted with a solid line.

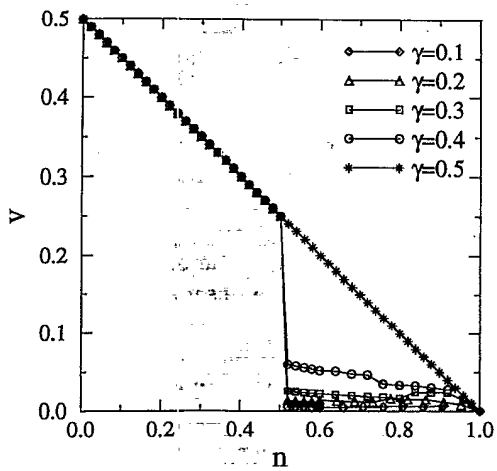


FIG. 5. Average velocity of cars, $v$, as obtained from the simulation of the nonlinear Boltzmann equations (12), vs the car density $n$. The transition to the jammed phase occurs at $n = 1/2$, regardless of the value of the turning parameter $\gamma$, in contrast with the microscopic simulations (compare with Fig. 1).

obtained, as well as the behavior of the remnant velocities after the transition has taken place.

In summary, the Boltzmann approximation gives good predictions for many physical quantities of the model, such as remnant velocity, size of the bands, onset time of instabilities, and predicts the existence of a jamming transition, although it does not capture the fact that the transition is $\gamma$ dependent. But with this simple scheme one can obtain a qualitatively correct picture of the behavior of the system.

## VI. A PHENOMENOLOGICAL MODEL

We have seen in Secs. III and V, respectively, an exact microscopic and an approximate Boltzmann description of the model. Both of them consider space-time variables that are discrete. In this section we are going to present a continuous phenomenological approach. Our motivation is twofold. In the first place it provides an alternative theoretical tool to understand the dynamics of the system. In the second place it makes a connection with the most usual approaches to traffic flow problems: continuous fluid dynamics [2]. This section only tries to fix the main lines of a more complete and exhaustive study of this traffic model from the point of view of continuous dynamics that will be presented elsewhere.

### A. Equations

To define the continuous model we have to consider first the important scales in the system, then redefine the variables, and finally take the appropriate limit. Let us assume that the distance between city crossings is $\epsilon$, that the linear size of the system is $\mathcal{L} \equiv L\epsilon$, and that there is a scale of speed for the cars $c$, defined as the speed of a car that moves freely (that is, the length it will travel, if it does not find any red light or other cars, divided by the spent time). The time interval taken by half a cycle of a traffic light (one time step) is then $\epsilon/c$.

Let us define a new set of space-time variables: $x \equiv n_x\delta$, $y \equiv n_y\delta$, and $\tau \equiv n_\tau\delta/c$, where $\delta \equiv l\epsilon \ll 1$ is a coarse-grain length scale ($l$ is an integer such that $1 \ll l \ll L$) we will define shortly, and $n_x, n_y, n_\tau$ are positive integers in the interval $[0, L/l]$. Now we introduce, as the variables of our model, two density functions, one for each type of car: $u(x,y;\tau)$ and $w(x,y;\tau)$, representing block averages in space and time of the Boolean variables $\mu_{\mathbf{r}}^t$ and $\nu_{\mathbf{r}}^t$:

$$u(x,y;\tau) \equiv \frac{1}{l^3} \sum_{(\mathbf{r},t)\in C_{(x,y)}\times I_\tau} \mu_{\mathbf{r}}^t \,, \tag{25}$$

$$w(x,y;\tau) \equiv \frac{1}{l^3} \sum_{(\mathbf{r},t)\in C_{(x,y)}\times I_\tau} \nu_{\mathbf{r}}^t \,, \tag{26}$$

where $C_{(x,y)}$ is a square of size $\delta\times\delta$ ($l^2$) sites, centered at $(x,y) \equiv (n_x\delta, n_y\delta)$ and $I_\tau$ is a time interval consisting of $l$ traffic light half cycles (i.e., it lasts a time $\delta_t \equiv \delta/c$). Notice that by definition the densities $u$ and $w$ are positive and less than 1:

$$0 \le u(x,y;\tau)\,, \quad w(x,y;\tau) \le 1\,, \tag{27}$$

and the dynamics, through the impossibility of having two cars at the same site, imposes

$$S(x,y;\tau) \;\le\; 1, \tag{28}$$

where we have defined the new variable $S(x,y;\tau) \equiv u(x,y;\tau) + w(x,y;\tau)$, which stands for the total density of cars.

Now let us take the continuous limit by making $n_x, n_y, n_\tau \to \infty$, $\epsilon \to 0$, and $l, L \to \infty$ while keeping $\delta \to 0$ and $(x,y) \in \Omega$, $\tau \in \mathbb{R}^+$, and $\mathcal{L}$ finite, where

$$\Omega = \{(x,y) : 0 < x, y < \mathcal{L}\} \in \mathbb{R}^2.$$

In this way the functions $u(x,y;\tau)$ and $w(x,y;\tau)$ represent coarse-grained local densities of the two types of cars. Because of this definition these density functions lose track of the microscopic details of the model (at scales $\epsilon$), and they are supposed to be almost constant up to scales of order $\delta$.

Let us now write the equations that govern the evolution of $u(x,y;\tau)$ and $w(x,y;\tau)$. We do so by writing a flux-balance equation for the cars of each type going in and out of a block $C_{(x,y)}$ that contains $l^2 \gg 1$ sites, with densities (constant by definition) $u(x,y;\tau)$ and $w(x,y;\tau)$. The cars going in are located either in the block $C_{(x-\delta,y)}$, inmediately to the left, or in the block $C_{(x,y-\delta)}$ inmediately below. The cars going out will go to the blocks $C_{(x+\delta,y)}$, located to the right, or to the one above, $C_{(x,y+\delta)}$. Let us estimate how many cars enter $C_{(x,y)}$ from $C_{(x-\delta,y)}$ in a time interval $\delta_t$. In the first traffic light half cycle only cars in the column sited at the right border of $C_{(x-\delta,y)}$ have any chance to go into $C_{(x,y)}$. On average, the number of cars of both types in

that column are $u(x-\delta,y;\tau)l$ and $w(x-\delta,y;\tau)l$. These cars have the chance to cross the left border of $C_{(x,y)}$ and be in its leftmost column. This column is occupied, on average, by $S(x,y;\tau)l$ cars, leaving $[1-S(x,y;\tau)]l$ empty sites. The number of cars that actually go through depends on the relative positions of the cars on both sides of the line and on the randomness built into the car movements. For simplicity we assume that the distribution of cars on both sides of the border is the most probable one: that in which the cars are uniformly distributed along the line of length $\delta$. In this case we can assume that in the first time step of $\delta_t$, $\gamma u(x-\delta,y;\tau)[1-S(x,y;\tau)]l$ cars of type $\mu$ and $\overline{\gamma}w(x-\delta,y;\tau)[1-S(x,y;\tau)]l$ of type $\nu$ will go from $C_{(x-\delta,y)}$ to $C_{(x,y)}$. During this first time step there has also been movement of cars inside $C_{(x-\delta,y)}$, but we are assuming that the densities $u(x,y;\tau)$ and $w(x,y;\tau)$ are constant in space, throughout the whole element, and in a time interval $\delta_t$. We will see later that this assumption is consistent with the resulting equations. Under this condition we can suppose that the rightmost column of $C_{(x-\delta,y)}$, and the leftmost column of $C_{(x,y)}$, are always occupied by the same number of cars. Therefore the total number of cars of type $u$ that will cross during $\delta_t$ ($l$ time steps) the left border of $C_{(x,y)}$ is

$$N^u_{\text{in}\rightarrow} = \gamma\, u(x-\delta,y;\tau)\, l\, [1-S(x,y;\tau)]\, l. \tag{29}$$

In the same way, the number of cars of type $u$ that will go inside $C_{(x,y)}$ from the block below, $C_{(x,y-\delta)}$, is

$$N^u_{\text{in}\uparrow} = \overline{\gamma}\, u(x,y-\delta;\tau)\, l\, [1-S(x,y;\tau)]\, l, \tag{30}$$

with the accompanying equations for $N^w_{\text{in}\rightarrow}$ and $N^w_{\text{in}\uparrow}$ exchanging $u$ and $w$ and $\gamma$ and $\overline{\gamma}$. Similarly, we can obtain expressions for the number of cars going out of $C_{(x,y)}$. Putting all together we can write the balance equation for the increment in the number of cars of each type in $C_{(x,y)}$:

$$\begin{aligned} 2l^2&[u(x,y;\tau+\delta_t)-u(x,y;\tau)] \\ &= N^u_{\text{in}\rightarrow} + N^u_{\text{in}\uparrow} - N^u_{\rightarrow\text{out}} - N^u_{\uparrow\text{out}} \\ &= l^2\,[\gamma\, u(x-\delta,y;\tau) + \overline{\gamma}\, u(x,y-\delta;\tau)][1-S(x,y;\tau)] \\ &\quad -l^2\, u(x,y;\tau)\,[1-\gamma\, S(x+\delta,y;\tau) - \overline{\gamma}\, S(x,y+\delta;\tau)] \end{aligned} \tag{31}$$

and the corresponding expression for $w$. The factor 2 comes from considering that only cars from one of the sides can enter $C_{(x,y)}$ at each traffic light half cycle. Observe that if we suppose that the spatial variables in Eq. (12) are continuous, we get exactly Eq. (31). In this way we can think of the coarse-grain treatment in this section as a continuous version of the Bolztmann approximation of Sec. V. Taylor expanding Eqs. (31) to first order in $\delta$ and $\delta_t$ and fixing $c=1$ to be consistent with the discrete model, we get the equation for the time evolution of $u(x,y;\tau)$ and $w(x,y;\tau)$:

$$u_\tau = \tfrac{1}{2}\{\gamma\,[u(S-1)]_x + \overline{\gamma}[u(S-1)]_y\}\ , \tag{32a}$$

$$w_\tau = \tfrac{1}{2}\{\overline{\gamma}[w(S-1)]_x + \gamma[w(S-1)]_y\}\ , \tag{32b}$$

where the subscripts $\tau$, $x$, and $y$ mean, respectively, $\partial/\partial\tau, \partial/\partial x$, and $\partial/\partial y$. These equations have to be completed with the periodic boundary conditions in $\mathcal{L}$ and initial conditions.

If Eqs. (32) represent faithfully the original microscopic discrete traffic flow model they should fulfill a few necessary (though, of course, not sufficient) conditions: conservation of the number of cars, positivity of the densities, excluded volume, and the existence of stationary solutions describing the two observed phases in the simulations. Let us check all of them. First observe that Eqs. (32) satisfy the following conservation laws (constant number of cars of each type) for any $\tau\in\mathbb{R}^+$:

$$\int_\Omega u(x,y;\tau)dxdy \;=\; \int_\Omega w(x,y;\tau)dxdy \;=\; n\mathcal{L}^2/2. \tag{33}$$

Second, both densities of cars have to stay positive during the evolution if the initial conditions were positive, and also the total number of cars per site should be less than 1 (excluded volume) (28). It can be shown from a maximum principle [21] that both restrictions hold during the evolution if the initial data $u(x,y;0)$ and $w(x,y;0)$ satisfy them. And, finally, from numerical simulations it is expected that the density functions $u$ and $w$ corresponding to a stationary uniform state will be solutions of Eqs. (32). It is also expected that a function of band type [see Fig. 1(b)], representing the jammed state, will be a solution. It can be inmediately seen that the uniform solution

$$\begin{aligned} u(x,y;\tau) &= n/2, \\ w(x,y;\tau) &= n/2 \end{aligned} \tag{34}$$

is stationary. It can also be checked that a jammed band configuration (of course parallel multiple bands are also possible)

$$\begin{aligned} u(x,y;\tau_0) &= \theta(x-x_0-y)\theta(a+y-x+x_0), \\ w(x,y;\tau_0) &= \theta(x-x_0+a-y)\theta(y-x+x_0), \end{aligned} \tag{35}$$

where $\theta(x)$ is the Heaviside step function, is a stationary solution too, but in the sense of the distributions [22]. Observe that the arguments of $\theta$ should be taken mod$\mathcal{L}$ due to the periodicity of the city. The arbitrary parameter $x_0$ fixes the position of the band and $2a = n\mathcal{L}$ is its width.

### B. Stability

We will proceed now to perform a stability analysis similar to that in Sec. V A. To study the linear stability of the solution (34) let us perturb around the uniform solution:

$$u(x,y;\tau) = n/2 + \delta u(x,y;\tau), \tag{36a}$$

$$w(x,y;\tau) = n/2 + \delta w(x,y;\tau), \tag{36b}$$

where $\delta u, \delta w \ll n$. Substituting in (32) we get, to first order

$$2\,P_\tau = \mathcal{A}\,P_x + \mathcal{B}\,P_y, \tag{37}$$

where

$$P \equiv \begin{pmatrix} \delta u \\ \delta w \end{pmatrix}, \quad \mathcal{A} \equiv \begin{pmatrix} \gamma\left(\frac{3}{2}n-1\right) & \gamma\frac{n}{2} \\ \overline{\gamma}\frac{n}{2} & \overline{\gamma}\left(\frac{3}{2}n-1\right) \end{pmatrix}, \tag{38}$$

and the expression for $\mathcal{B}$ is the same as the one for $\mathcal{A}$ exchanging $\gamma$ and $\overline{\gamma}$. Expanding in its Fourier series, Eq. (37) can be easily integrated. If we define the Fourier coefficients

$$\widehat{P}(k_x, k_y; \tau) = \int_\Omega \frac{dxdy}{\mathcal{L}^2}\, P(x_+, x_-; \tau)\, e^{-ik_x x - ik_y y}, \tag{39}$$

with $k_{x(y)} = 2\pi n_{x(y)}/\mathcal{L}$, $n_{x(y)} \in \mathbb{Z}$, (37) becomes

$$2\,\widehat{P}_\tau = ik_x\mathcal{A}\,\widehat{P} + ik_y\mathcal{B}\,\widehat{P}, \tag{40}$$

whose solution is

$$\widehat{P}(k_x, k_y; \tau) = e^{i\frac{\tau}{2}(k_x\mathcal{A}+k_y\mathcal{B})}\,\widehat{P}(k_x, k_y; 0). \tag{41}$$

The eigenvalues of $k_x\mathcal{A} + k_y\mathcal{B}$ are

$$\lambda_{1,2} = k_+\left(\tfrac{3}{2}n - 1\right) \pm\sqrt{2\zeta^2 k_-^2(1-n)\left(\tfrac{1}{2}-n\right) + k_+^2\frac{n^2}{4}}, \tag{42}$$

where $k_\pm \equiv (k_x \pm k_y)/2$. The homogenous solution (34) will become unstable if the term inside the square root in (42) becomes negative, that is, if

$$2\zeta^2 k_-^2(1-n)\left(\tfrac{1}{2}-n\right) + k_+^2\frac{n^2}{4} < 0. \tag{43}$$

If $n > \frac{1}{2}$ condition (43) is always fulfilled, for any value of $k_+$, for some big enough value of $k_-$. Once again we obtain that the uniform solution becomes unstable for $n > \frac{1}{2}$. Observe that Eq. (43) agrees to first order with the result given in Eq. (22) when we make $(k_x, k_y) = (-k/\sqrt{2}, k/\sqrt{2})$. This is because Eq. (40) is just the continuous version of Eq. (17) to order $k$, showing again the correspondence between both descriptions [see also the comments below Eq. (31)].

### C. Velocity

Following the reasoning of Sec. VI A we can also get a phenomenological estimate of the instantaneous average velocity in this model.

The velocity is proportional to the number of movements that take place in the system at any given time. We can measure the total number of produced movements by estimating how many cars cross horizontal and vertical line elements of length $\delta$. For example, the numbers of cars of type $u$ and $w$ crossing a vertical line of length $\delta$ centered at $(x, y)$ in a time step are [see Eq. (29) for a similar reasoning], respectively, $u(x - \epsilon/2, y; \tau)l[1 - S(x + \epsilon/2, y; \tau)]\gamma$ and $w(x - \epsilon/2, y; \tau)l[1 - S(x + \epsilon/2, y; \tau)]\overline{\gamma}$ where we have explicitly pointed out that the cars crossing the line are just *half* a site to its left and they move to a place half a site to its right, and therefore the corresponding densities should be evaluated at $x - \epsilon/2$ and $x + \epsilon/2$, respectively. Similarly, the numbers of cars of type $u$ and $w$ crossing a horizontal line of length $\delta$ centered at $(x, y)$ are [see (30)], respectively, $u(x, y - \epsilon/2; \tau)l[1 - S(x, y + \epsilon/2; \tau)]\overline{\gamma}$ and $w(x, y - \epsilon/2; \tau)l[1 - S(x, y + \epsilon/2; \tau)]\gamma$. But when we take the limit $\epsilon \to 0$ we can replace $u(x \pm \epsilon/2, y; \tau)$ by

$$u(x^\pm, y; \tau) \equiv \lim_{\epsilon\to 0+} u(x \pm \epsilon/2, y; \tau), \tag{44}$$

and $u(x, y \pm \epsilon/2; \tau)$ by

$$u(x, y^\pm; \tau) \equiv \lim_{\epsilon\to 0+} u(x, y \pm \epsilon/2; \tau). \tag{45}$$

Taking care of all the terms in the same way, properly normalizing (the factor 1/2 comes from the traffic lights), and taking the limit $\epsilon \to 0$, $L \to \infty$, $\mathcal{L}$ finite, we get the total speed of the system as

$$\begin{aligned} v = \frac{1}{2n\mathcal{L}^2}\int_\Omega dxdy\{&[1 - S(x, y^+; \tau)] \\ &\times[\overline{\gamma}u(x, y^-; \tau) + \gamma w(x, y^-; \tau)] \\ &+[1 - S(x^+, y; \tau)][\gamma u(x^-, y; \tau)] + \overline{\gamma}w(x^-, y; \tau)\}. \end{aligned} \tag{46}$$

With this expression we can easily calculate the speed of the homogenous solution (34)

$$v = \frac{1-n}{2}, \tag{47}$$

in accord with all the previous models and the simulations. We can also calculate the speed for a solution of the jammed band type (35) getting, before taking the limit inside the integral,

$$v = \frac{\gamma}{\mathcal{L}n}\epsilon = \lim_{L\to\infty}\frac{\gamma}{Ln}. \tag{48}$$

When we take the limit $\epsilon \to 0$ (keeping $\mathcal{L}$ and $n$ finite) we get $v = 0$. This result can be easily understood. In one band the only possible movements are along its border ($\sim \gamma L$) and therefore the contribution to the velocity is of order $\gamma L/nL^2$. This is exactly Eq. (48). This leads us to conclude that an infinite number of bands is necessary to have nonzero velocity in the jammed phase of an infinite system. For example, if there is a band every $\lambda$ sites [see the discussion below Eq. (21)], the velocity will be of order

$$\frac{\gamma}{nL}\frac{L}{\lambda} = \frac{\gamma}{n\lambda}, \tag{49}$$

which is finite in the $L \to \infty$ limit.

Let us return to Eq. (48). Though it is zero in the $\epsilon \to 0$ limit it gives a prediction on the behavior of the $v$-$n$ curves after the transition when $L$ is finite and $\epsilon > 0$,

$$v = \frac{\gamma}{nL}. \tag{50}$$

To compare this prediction with the results of simula-

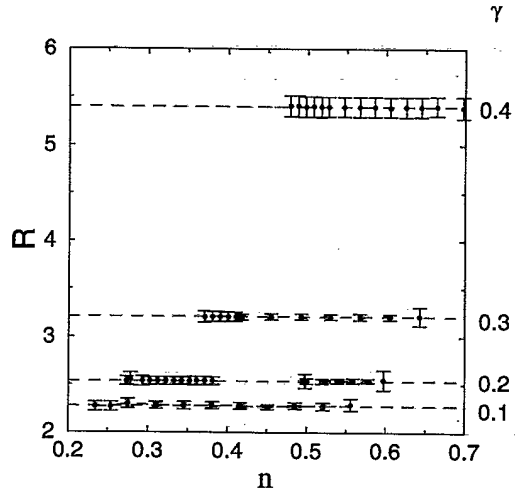


FIG. 6. $R \equiv \bar{v}Ln/\gamma$ against the car density $n$ for the data taken from Fig. 1 in the single-band jammed phase [see discussion after Eq. (50)].

tions we have taken data from Fig. 1. We have considered only the points that represent stationary final states with one band [6] and we have plotted in Fig. 6 the ratio $R \equiv Lnv/\gamma$ against $n$ for four different values of $\gamma$. If the prediction of Eq. (50) were completely correct the four graphs should be the horizontal lines $R = 1$. Instead, as can be seen in the figure, the data fit well to four horizontal lines with different values of $R$. The first important conclusion to draw from Fig. 6 is that, because the data fit to horizontal lines, Eq. (50) gives the correct behavior of the velocity after the transition: it decreases with $n$ as $1/n$ and is proportional to $\gamma/L$. The discrepancy is in the proportionality factor, and may be due to the fact that the jammed phase obtained in the simulations is not a "discontinuous" band (that goes sharply from 1 to zero in a distance $\epsilon$) as is the one in Eq. (35). Observing carefully the final configurations it can be seen that in fact the intermeditate region where the density goes from 1 to zero, though narrow, extends for several lattice spacings, being narrower the smaller is $\gamma$ (the system is less random). The values of $R$ in Fig. 6 confirm this behavior too.

## VII. CONCLUSIONS

In this paper we have developed a theoretical framework to describe the two-dimensional cellular automata traffic flow model introduced in Ref. [6]. The starting point is the exact microscopic evolution equations (1) together with the microscopic definition of the mean velocity (4). From these basic ingredients several approximate schemes can be carried out. The simplest one is to follow the standard kinetic recipe of breaking up spatial correlations into products of one-particle averages. This mean-field-like approach leads to a reasonably good description of the model. In the first place, the $\gamma$-independent (for $\gamma > 0$) slope of the $v$-$n$ curves in the freely moving phase is correctly obtained. In the second place, the existence of a phase transition between the freely moving and the jammed states is also explained by such a simple approximation. Nevertheless, it fails to describe properly this transition because it predicts a $\gamma$-independent transition density. Finally, this theory allows us to understand the rich structure of the jammed phase. In this phase, the Boltzmann approximation is able to explain the formation of bands completely filled with cars, and the phenomenon of phase separation of the two populations of cars inside these bands. Besides, it gives an answer to the question posed by the simulations about the number of bands in the jammed state. Our Boltzmann theory gives an estimation of the average band-to-band distance. This distance appears to be finite for all densities in the range $1/2 < n < 1$ (excluding the transition density according to this approximation, $n = 1/2$, and the full system $n = 1$). The conclusion is that there is an infinite number of bands in an infinite system. On the other hand, the band-to-band distance estimation let us evaluate the most probable number of bands for finite $L$, once the values of $\gamma$ and $n$ are fixed. The results obtained are in good agreement with the simulations.

In Sec. VI we have faced the description of the model using an utterly different phenomenological approach. This approach ends up in a continuous model whose evolution is governed by two partial differential equations. Although this approach seems to have no relation to the Boltzmann approach, we have shown that there is indeed a connection to the kinetic formalism—from which it can be obtained in the large spatial scale regime. The continuous approximation leads to a connection between our model and those of fluid dynamics—the most usual tools to study one-dimensional traffic flow problems. Apart from its theoretical relevance, this phenomenological model is able to provide a $\gamma$-dependent expression for the velocity as a function of $n$ in the jammed phase. This result, together with the linearly decreasing velocity obtained for the freely moving phase, completes a theoretical description of the $v$-$n$ phase diagram.

The theory presented in this paper for a particular model opens new ways in the theoretical study of two-dimensional traffic flow, which deserve future development. First of all, it should be clear that equations similar to (1) can be written for any cellular-automaton-based traffic model and, therefore, the same approximations can be applied to get information out of it. It is also feasible to write down a continuous model, either by itself or as a limit of the kinetic description. Secondly, it is possible to devise other approximations based on the microscopic equations (1). For instance, it is our belief that the next step in this scheme—namely, including two-point spatial correlations—might account for the $\gamma$ dependence of the transition density. We are presently working on it. Finally, it is worth remarking that the approximate models reported in this work are far from being exhausted. Questions such as the mathematical properties of the instabilities in the continuous model or how to get an evolution equation for the velocity remain unsolved.

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[10] In principle, by virtue of the traffic lights, which introduce a time inhomogeneity, there might be two different steady states: one for even and one for odd time steps. In that case, Eq. (5) would be the average of the mean velocity in those two steady states.

[11] In Ref. [6] this contradiction led us to suggest that the transition in Ref. [5] might fall to $n = 0$ in an infinite system. With the new analysis this claim can no longer be made and so it remains an open question.

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