ABSENCE OF DISSIPATIVE SOLUTIONS OF THE SCHRÖDINGER AND KLEIN-GORDON EQUATIONS WITH LOGARITHMIC NONLINEARITY

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It is shown that neither the Schrödinger equation nor the Klein–Gordon one with logarithmic nonlinearities have dissipative solutions. In the case of one-dimensional space, numerical experiments with different Cauchy data, in the nonrelativistic case, lead always to final states consisting only in oscillating gaussons.

1. Introduction

In 1976 [1] Bialynicky-Birula and Mycielski (BBM) proposed the study of the Schrödinger equation with logarithmic nonlinearity

$$\mathrm{i}\frac{\partial\psi}{\partial t}=\left(-\frac{1}{2m}\Delta-b\log(|\psi|^2a^n)\right)\psi\,, \qquad (1)$$

where a and b are constants and n is the number of dimensions of the space. BBM showed that (1) is the only equation on which a nonlinear form of quantum mechanics can be based, without substantial changes in the interpretation of the wavefunction. It has solitonlike solutions with gaussian shape (they accordingly called them gaussons) in any number of dimensions, which describe the propagation of nonspreading wave packets of freely moving particles. Soon after Oficjalski and Bialynicki-Birula [2] performed numerical experiments which showed that the gaussons have real solitonlike behaviour upon collision, as two of them give a final state consisting either of two or three gaussons depending on the relative velocity and phase.

BBM suggested that their equation could be applied to atomic or molecular physics and Shimony [3] proposed an experiment with neutron interferometry to look for the effects associated with the nonlinearity. However the experiments realized [4,5] placed very strong limits on the constant b and are currently interpreted as indicating that there is no real basis for such a nonlinearity.

In spite of this negative experimental result, Hefter [6] argued recently that it may be suitable to describe extended objects (and not point particles as originally suggested by BBM). Consequently, he applied it to nuclear physics and showed that it can account for some of the properties of nuclear matter.

The BBM equation has also attracted the attention of mathematicicians, who studied such properties as the Cauchy problem, asymptotic behaviour and stability of gaussons [7–9].

2. Nonexistence of dissipative solutions

Let $\psi(\boldsymbol{r},t)$ be a solution of (1) and let $M(t)$ be its norm L^∞, that is

$$M(t)=\sup_{r}|\psi(\boldsymbol{r},t)|\,. \qquad (2)$$

It is said that the solution $\psi(\boldsymbol{r},t)$ is dissipative or that it dissipates if

$$\lim_{t\to\infty}M(t)=0\,. \qquad (3)$$

If, on the contrary, $M(t)$ is bounded from below, the solution is said to be nondissipative. In 1978 Morris [10] proved in the relativistic case that a class of

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nonlinear Klein–Gordon equations (including one with a fractional nonlinearity proposed by Werle [11] do not have dissipative solutions. However, as he himself states, his proof does not apply to the logarithmic Klein–Gordon one. Neither does it to the nonrelativistic BBM equation. However, it follows from Cazenave's results (proposition 4.3 of ref. [8]) that this last one has the same property. We will now show that this is indeed the case in a simpler, less formal and more physically oriented way. The proof, although different, is similar to the one used by Morris.

The lagrangian and the energy densities, L and H, of (1) are

$$L=\tfrac{1}{2}\mathrm{i}[\psi^*\partial_t\psi-(\partial_t\psi^*)\psi]-\frac{1}{2m}\nabla\psi^*\cdot\nabla\psi$$
$$+b\log(\psi^*\psi a^n)\psi^*\psi-b\psi^*\psi\,, \tag{4}$$

$$H=\frac{1}{2m}\nabla\psi^*\cdot\nabla\psi-b\log(\psi^*\psi a^n)\psi^*\psi+b\psi^*\psi\,. \tag{5}$$

Unless otherwise stated, by the norm of a solution we will understand the norm L^2, which happens to be conserved

$$\|\psi\|=\int\limits_{\mathbb{R}^3}|\psi|^2\,\mathrm{d}^3r=\text{constant}\,, \tag{6}$$

Let us now note that if $\psi(\boldsymbol{r},t)$ is a solution, $\alpha\psi(\boldsymbol{r},t)\exp(ibt\log|\alpha^2|)$ is another one and, because of that, we can assure that, if all solutions which have unit norm are nondissipative, all the rest have the same property, notwithstanding the value of their norms. For if ψ' with norm β^2, β being a positive real number, is a dissipative solution, the same can be said of $\psi=\beta^{-1}\psi'\exp(-2ibt\log\beta)$ which evidently has unit norm. It suffices, therefore, to prove that no unit norm solution dissipates.

The energy of a unit norm solution can be written as

$$E=\frac{1}{2m}\int|\nabla\psi|^2\,\mathrm{d}^3r+\int U(\psi^*\psi)\,\mathrm{d}^3r$$
$$+b(1-\log a^n)\,,$$

where the function U has the following properties:

(a) $U(\rho)=-b\rho\log\rho$; $U'(\rho)=-b\log\rho-b$; $U''(\rho)=-b/\rho$;

(b) U is concave in $(0,\infty)$ and $\lim U'(\rho)=+\infty$, if $\rho\to 0$ and

(c) $U(\rho)\geqslant\rho U'(\rho)$.

Let us define $a(t)$ as

$$a(t)=\int\limits_{\mathbb{R}^3}U(\psi^*\psi)\,\mathrm{d}^3r\,.$$

It is clear that

$$E^*=E-b(1-\log a^n)\geqslant a(t)$$

(if the energy is finite, the sign $>$ always holds). As

$$|\psi(\boldsymbol{r},t)|^2\leqslant M^2(t)$$

and, because $U'(\rho)$ is a decreasing function, it turns out that

$$U'(|\psi|^2)\geqslant U'(M^2),$$

after which, it follows from (c) that

$$U(|\psi|^2)\geqslant U'(|\psi|^2)|\psi|^2\geqslant U'(M^2)|\psi|^2$$

and, integrating in all the space

$$E^*\geqslant\int\limits_{\mathbb{R}^3}U(\psi^*\psi)\,\mathrm{d}^3r\geqslant U'(M^2)\,.$$

If we assume the solution to be dissipative, $M(t)\to 0$, if $t\to\infty$, the right-hand side tending to ∞, which is impossible since E^* is conserved. The consequence: the modulus of $\psi(\boldsymbol{r},t)$ is necessarily bounded from below.

In the relativistic case, it can be shown that all finite energy solutions verifying $0<|Q|<\infty$ are nondissipative. Let the lagrangian density and wave equation be

$$L=(\partial_\mu\phi^*)\partial^\mu\phi-W(\phi^*\phi)\,,$$

$$W(\rho)=-b\rho\log\rho+m^2\rho\,, \tag{7}$$

$$\partial_\mu\partial^\mu\phi+W'(\phi^*\phi)\phi=0\,. \tag{8}$$

The corresponding energy and charge are then

$$E=\int[|\phi_t|^2+|\nabla\phi|^2+W(|\psi|^2)]\,\mathrm{d}^3r\,, \tag{9}$$

$$Q=\mathrm{i}\int(\phi^*\phi_t-\phi_t^*\phi)\,\mathrm{d}^3r\,, \tag{10}$$

W having the same properties (b), (c) as in the nonrelativistic case. Let ϕ be a solution of (7) with finite energy and non-null finite charge. From (10) it follows that

$$|Q| \leqslant 2 \int |\phi\phi_t| \, \mathrm{d}^3r \tag{11}$$

and, after application of the Schwarz inequality,

$$\tfrac{1}{4}Q^2 \leqslant \left(\int |\phi_t|^2 \, \mathrm{d}^3r \right) \left(\int |\phi|^2 \, \mathrm{d}^3r \right). \tag{12}$$

Making use now of property (c) of U and of the fact that U' is strictly decreasing

$$E \geqslant \int |\phi_t|^2 \, \mathrm{d}^3r + \int W'(|\phi|^2)|\phi|^2 \, \mathrm{d}^3r$$
$$\geqslant \int |\phi_t|^2 \, \mathrm{d}^3r + W(M^2) \int |\phi|^2 \, \mathrm{d}^3r, \tag{13}$$

where $M(t)$ is defined by (2). If the solution is dissipative

$$\lim M(t) = 0$$
$$\Rightarrow \exists t_0 / \forall t > t_0, \quad W'(M^2) > \delta > 0 \quad \text{for some } \delta,$$

from which, because of (13)

$$E \geqslant \int |\phi_t|^2 \, \mathrm{d}^3r \tag{14}$$

and from (12)

$$\int |\phi|^2 \, \mathrm{d}^3r \geqslant Q^2/4E. \tag{15}$$

On the other hand, the definition of E and the properties of $W(\rho)$ imply

$$E \geqslant \int_{t>t_0} W(|\phi|^2) \, \mathrm{d}^3r \geqslant \int W'(|\phi|^2)|\phi|^2 \, \mathrm{d}^3r$$
$$\geqslant W'(M^2) \int |\phi|^2 \, \mathrm{d}^3r > 0$$

and, from (15)

$$4E^2/Q^2 \geqslant W'(M^2) > 0. \tag{16}$$

If $\lim M(t) = 0$ when $t \to \infty$, then $\lim W'(M(t)) = \infty$, in clear contradiction of (16). The conclusion: ϕ must be nondissipative.

3. Numerical experiments

The previous result agrees with the conjecture stated by Oficjalski and Bialynicki-Birula [2] in the nonrelativistic case, according to which "every nonlinear wave described by the logarithmic equation eventually decays into (possibly oscillating and rotating) gaussons". In order to test this conjecture in the case of one space dimension we performed several numerical experiments on the logarithmic Schrödinger equation with several classes of Cauchy data. First of all, we used the same method as in ref. [2] (taken from ref. [12]), reproducing the same results. However, as this scheme does not conserve the energy, one has to be very careful. For instance, let us consider the evolution of oscillating gaussons (henceforth to be called pulsons) of the form

$$\phi(x,t) = A(t) \exp[\mathrm{i}\phi(t)]$$
$$\times \exp\{-\tfrac{1}{2}[B(t) + \mathrm{i}C(t)]x^2\}, \tag{17}$$

which are solutions of (1) if

$$\frac{\mathrm{d}}{\mathrm{d}t}B = 4BC, \quad \frac{\mathrm{d}}{\mathrm{d}t}C = 2(bB - B^2 + C^2),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}A = AC, \quad \frac{\mathrm{d}}{\mathrm{d}t}\phi = 2b \ln A - B. \tag{18}$$

With the said numerical method, we found a tendency to dissipate the energy and tend towards a stationary gausson ($B(t) = b$). Consequently we used another more accurate one, based on the scheme

$$-\frac{1}{2m}\left(\frac{\psi_{j+1}^{n+1} - 2\psi_j^{n+1} + 2\psi_{j-1}^{n+1}}{2(\Delta x)^2} + \frac{\psi_{j+1}^n - 2\psi_j^n + \psi_{j-1}^n}{2(\Delta x)^2} \right)$$

$$-\frac{G(|\psi_j^{n+1}|^2) - G(|\psi_j^n|^2)}{|\psi_j^{n+1}|^2 - |\psi_j^n|^2} \frac{\psi_j^{n+1} + \psi_j^n}{2}$$
$$= \mathrm{i}\frac{\psi_j^{n+1} - \psi_j^n}{\Delta t},$$

with

$$G(\rho) = -b\rho(1 - \ln \rho). \tag{19}$$

This scheme conserves the norm and the energy but is nonlinear. In fig. 1 the results of the two schemes, as applied to the evolution of a pulson (17), are compared. In fig. 1a we can see the phase space (B, C). B and C are periodic functions of time and, as a consequence of (18), A is also periodic. In figs. 1b and 1c the evolution of $B(t)$, as calculated with the schemes of ref. [2] and (19), respectively, and with the same lattice in both of them is shown. As we see this second one gives a much better result.

We have used several classes of Cauchy data and

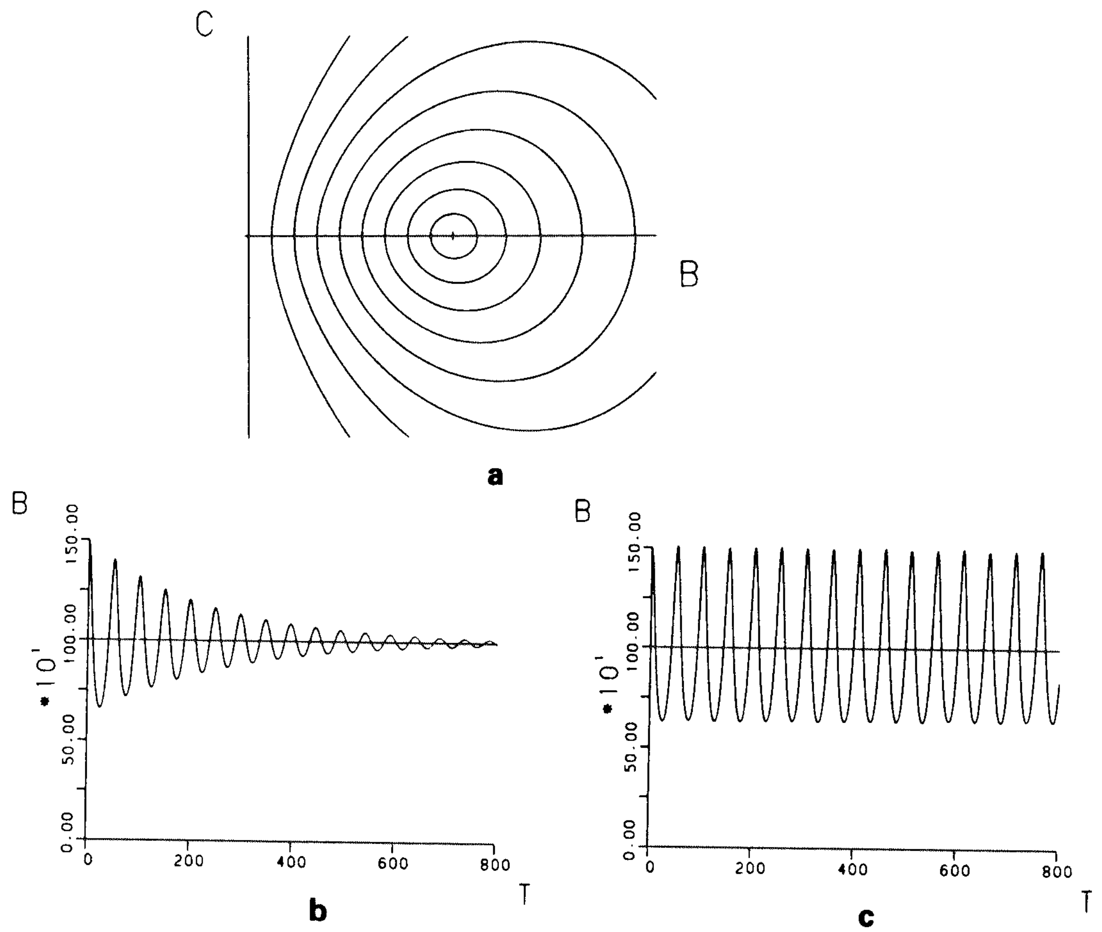

Fig. 1. Comparison of the results of the two methods as applied to a pulson. (a) Phase space of the pulson; (b) scheme of ref. [2]; (c) scheme (19).

found always the same kind of behaviour. The following two examples are representative of the rest:

(i) Collisions of two gaussons. We have observed, as in ref. [2], that there is a resonance energy interval in which the final state consists of three pulses. Outside of it, there are only two. But we have also been able to determine clearly that the pulses are not gaussons but pulsons and to calculate its parameters. In fig. 2 we can see the charge density $|\psi(x,t)|^2$ versus x for eight values of t. In order to ascertain the nature of the bumps, we fitted them to a pulson of the form (17) and determined the values of the parameters A and B which give the minimum quadratic error. The result for the central pulse can be seen in fig. 3. In fig. 3c the relative error of the fit is shown, as measured by the quadratic norm of the difference divided by that of $\psi(r,t)$. Its low value indicates that the pulse is indeed either a pulson or is very close to one.

(ii) Decay of an initial wavepacket. We have studied the evolution from several localized states, the results being similar in all the cases. In fig. 4 we show the case of the sine wave

$$\begin{array}{ll} \sin 2\pi x\,, & x \in |-\frac{1}{4}, \frac{1}{4}|\,, \\ 0\,, & x \notin |-\frac{1}{4}, \frac{1}{4}|\,, \end{array}$$

which decays into three pulses. In fig. 5 the result of a fit to a pulson, similar to that of fig. 3 can be seen. Again, the central packet is a pulson.

In agreement with the lack of radiation, the sum of the charges of the final pulsons is in all the cases equal to the initial charge and the same thing happens with the energy, the differences being smaller

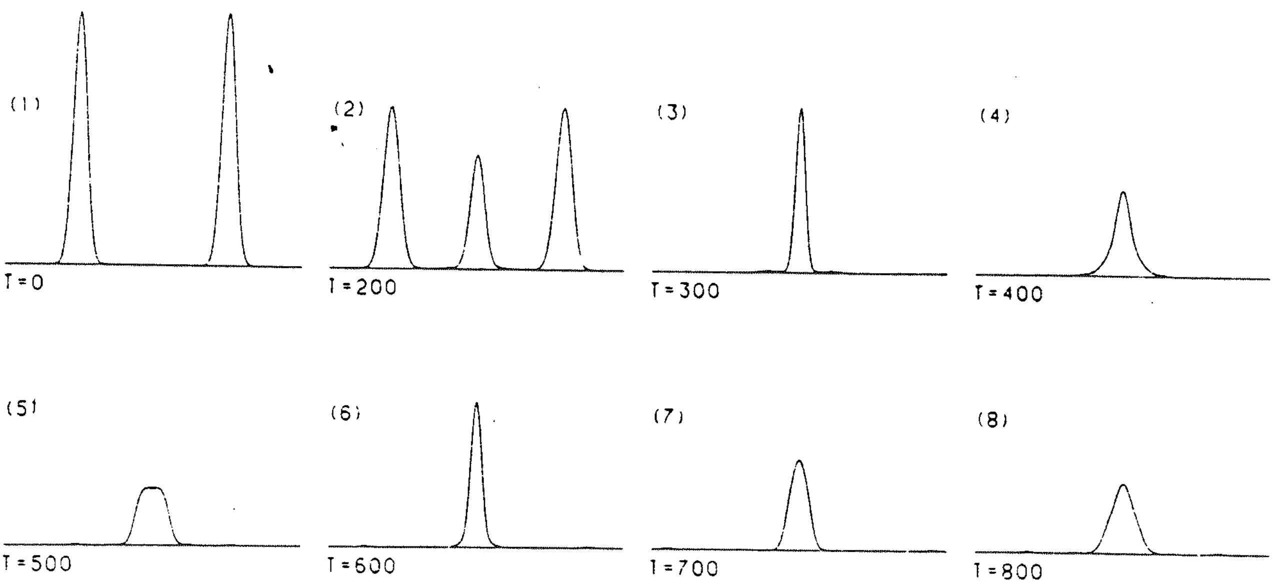

Fig. 2. Collision of two gaussons. (1) Initial data; (2) pulsons after the collision; (3)–(8) evolution of the central pulson after elimination of the two lateral ones.

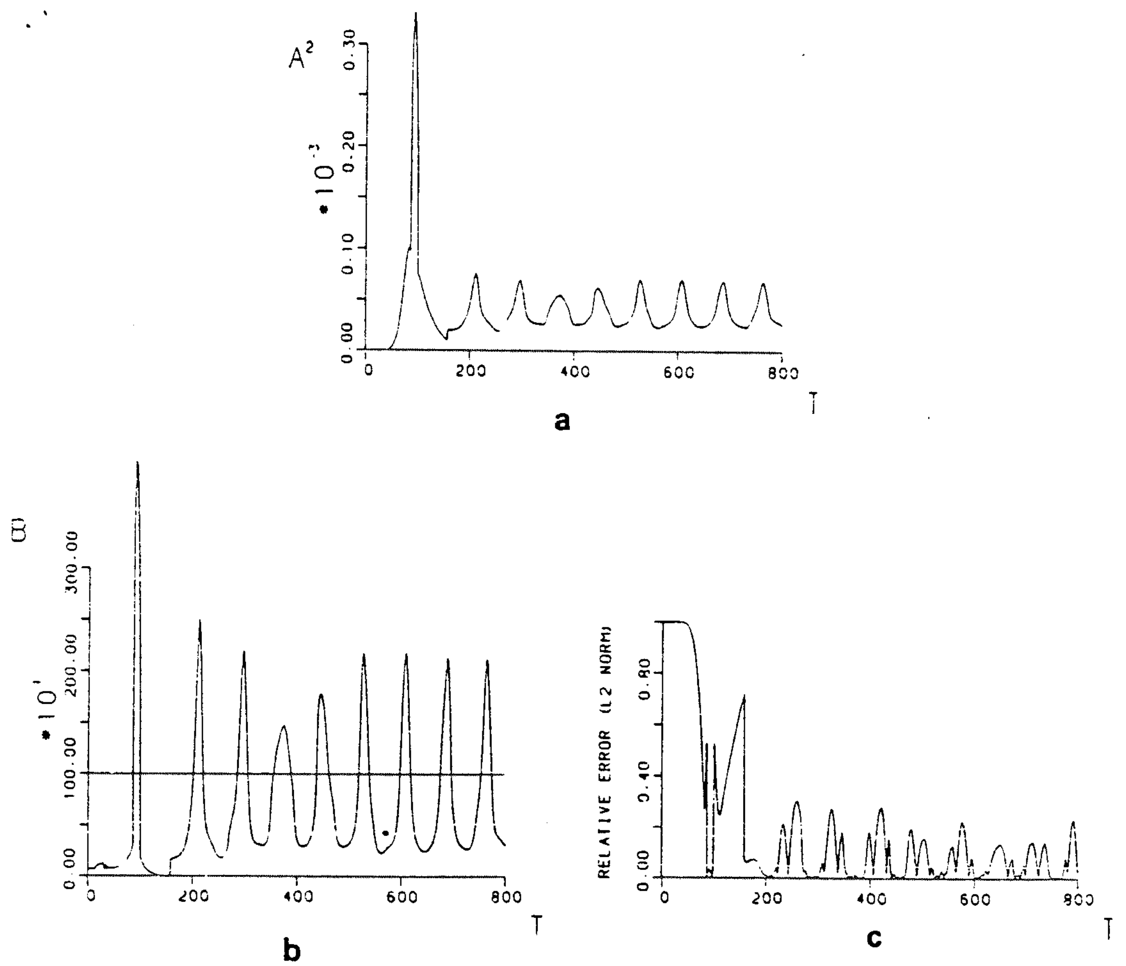

Fig. 3. Fit to a pulson of the central bump of fig. 2. The error is partly due to the effect of the elimination of the lateral bumps.

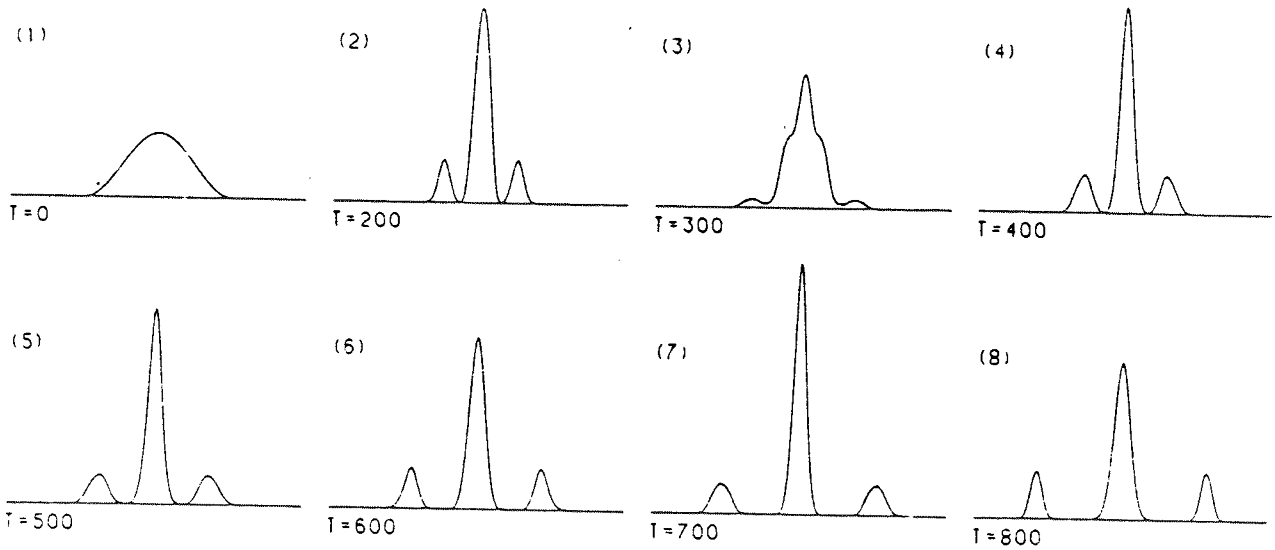

Fig. 4. Decay of a sine wave into three bumps.

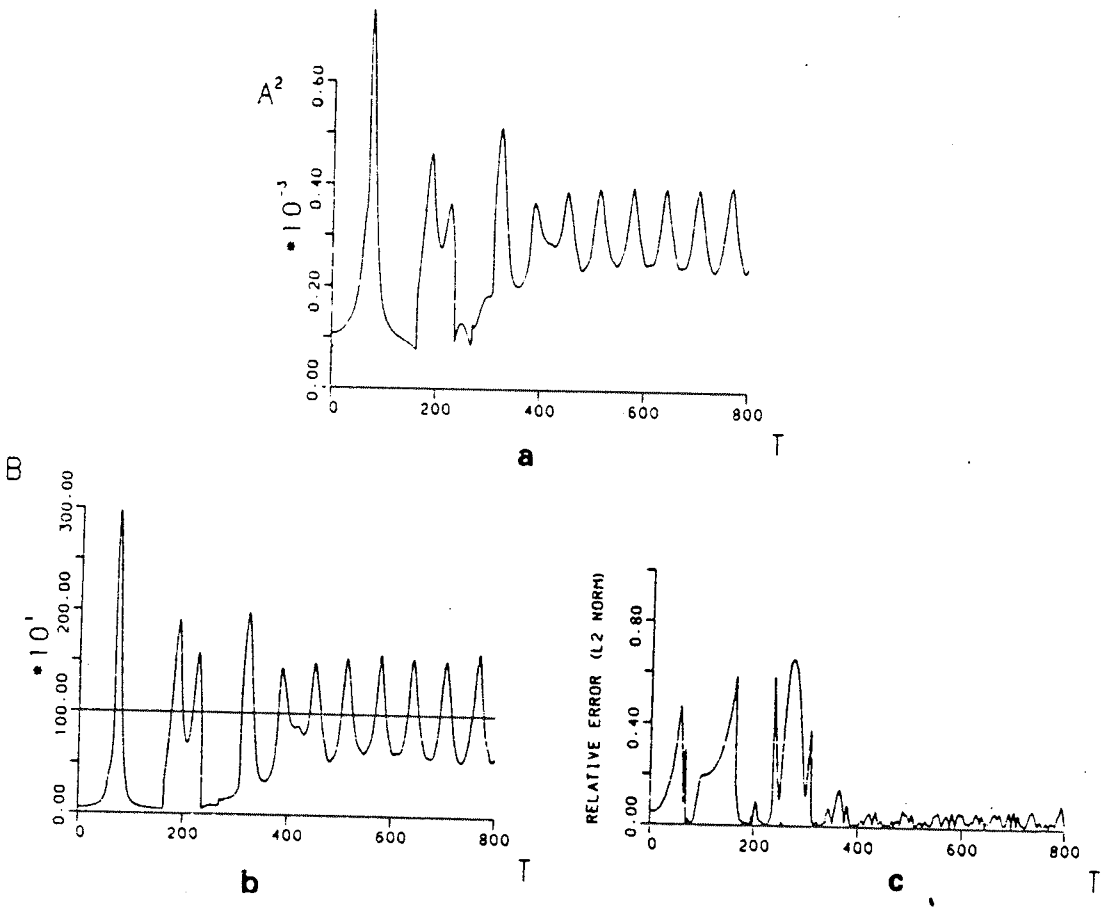

Fig. 5. Fit to a pulson of the central bump of fig. 4.

than 1% and compatible with zero, taking into account the numerical errors.

4. Conclusions

(1) The logarithmic Schrödinger equation proposed by Bialynicki-Birula and Mycielski has no dissipative solutions. In the relativistic case, the logarithmic Klein–Gordon equation has no dissipative solutions with nonzero charge.

(2) Our numerical results are compatible with the conjecture that any final state consists always of nothing more than oscillating gaussons. Moreover, we have determined the parameters of the pulsons which appear after the collision of gaussons or the decays of some initial data.

To sum up, we interpret the previous results as strongly supporting the idea that the wave equation with logarithmic nonlinearity deserves further study and consideration.

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